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A Quarterly Dedicated to the Service of Students and Teachers of Mathematics in India.

VOL. VII.—1939

Edited by

A. NARASINGA RAO, M.A., L.T.

Annamalai University, Annamalainagar, South India.

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THE BAKSHALI MANUSCRIPT

BY

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§ 1. Introductory

In spite of the publication by the Archæological Survey of India, of the text of the Bakshālī Manuscript † with elaborate introduction and notes by G. R. Kaye, there are perhaps many fresh points tending to throw additional light on the contents and the age of the manuscript. A considerable portion of the analysis of the manuscript was really due to Hoernle, and by an irony of fate, the work was left for completion to one whose views and prejudices were poles apart from those of the pioneer. It is regretted that Kaye's examination apparently scientific, is warped by his profound obsession that he should find traces of Greek influence in everything Indian. This attitude has vitiated most of his conclusions and made it worth while for others * to re-examine his statements and give them their proper weight. The present paper is an attempt at such re-examination and re-assessment.

It is now sufficiently known that a manuscript of 70 leaves of birch-bark, each 7″ by 4″, dealing mainly with Arithmetic was discovered in 1881 by a peasant, near a village called Bakshali about 70 miles from the famous Taxila in the north-western frontier of India. This manuscript has since been christened "The Bakshālī Manuscript." The text is written in Sarada script the date of which is highly disputed and the language is the so-called *Gatha* dialect a form of north-western *Prakrit*. Apart from the script and the language, there are many surprises in the mathematical contents

† The Bakshāli Manuscript by G. R. Kaye, *Archæological Survey of India*, New Imperial Series, Vol. XLIII, Parts I and II (1927) and Part III (1933). Hereafter referred to as "B. M. I, II, III."

* For example, *The Bakshāli Mathematics*, by Bibhutibhusan Datta, 1929; hereafter referred to as "Datta."

which make one hesitate in fixing its age. These surprises and peculiarities will now be studied under seven heads:

(i) The method of presentation,

(ii) Peculiar terminology, abbreviations, and the cross-symbol for subtraction,

(iii) The Decimal Notation and the absence of word-numerals,

(iv) The symbol for the unknown,

(v) The Rule of *Regula Falsi*,

(vi) The square-root rule and the process of reconciliation,

and (vii) Series and sequences.

§ 2. The Method of Presentation

Rules and examples are presented in verse in Sloka metre and the explanations are given in prose according to a certain stereotyped convention as follows:

(1) The re-statement of the problem in a symbolic notation—called Sthāpana or Nyāsa.

(2) The working proper—called Karaṇa

and (3) Verification—called Pratyaya.

This bears some analogy to the Euclidean scheme of

(1) particular enunciation corresponding to Nyāsa,

(2) construction corresponding to Karaṇa,

and (3) demonstration corresponding to verification.

It is interesting to note that among the early arithmeticians verification had played an important part and served almost as an equivalent of 'logical proof'. The majority of minds are content with the working rule, provided verification shows them that the results are correct. They do not care to worry themselves about the rationale. In actual teaching again, we find a pupil more easily convinced by verification than by a closely reasoned mathematical argument too difficult to follow. Thus, verification is a potent instrument in the pedagogy of mathematics. No wonder, therefore, that a mathematics teacher of bygone days should have thought of incorporating 'verification' as an act of faith in a work intended especially for the benefit of young pupils †. In the Bakshālī Manuscript we have perhaps a glimpse of a sort of teaching notes—something intermediary to an original treatise and a regular commentary—of a private tutor. The loose colloquial style adopted is also in keeping

† Vide the Colophon. B. M. II 50, recto. P. 142.

with this idea. One may also recollect in this connection that Bakshālī was near the famous ancient University centre Taxila.

'Verification' was common enough among the Indian mathematical commentators as early as the time of Chaturveda Prithūdakaswami of the 9th century A. D. The rule of Vyastavidhi or reversing the steps, discussed in almost all ancient Indian mathematical works is perhaps intended to be useful in 'verification', especially of the roots of equations and in the so-called 'think of a number' problems. For example,* we have the result $1+\frac{1}{2}\{2+\frac{1}{2}(3+\frac{2}{2}\cdot\overline{4+\frac{3}{2}\cdot\overline{5+\frac{4}{2}\cdot\frac{1}{4}}})\}=\frac{93}{16}$ verified by the principle of reversal of steps, viz.

$\frac{2}{4}\left[\frac{2}{3}\left[\frac{2}{2}\left[2\left\{2\left(\frac{93}{16}-1\right)-2\right\}-3\right]-4\right]-5\right]=\frac{1}{4}$, worked up from the innermost bracket outwards step by step.

§ 3. Peculiar Terminology etc.

The mathematical terminology adopted in the Bakshālī Manuscript though generally the same as in other Hindu Mathematical works, contains a few exceptions incidental more or less to the language or dialect employed. For example, the terms such as 'Partha', 'Dhanta', 'Pasta' are peculiar to the Gatha dialect. The use of the terms 'Savarna', 'Kalāsavarna', and 'Sadris'īkarana' with reference to fractions remind one of similar uses in other Hindu works.

There are several abbreviations, some of which are consistently employed, while others are used without any uniformity. For example, the abbreviation *yu* for *yutam* (for addition) is sometimes put in between and sometimes after the expressions to be summed:

** e.g. $\left|\begin{matrix} & 5 & yu^{\circ} \\ 1 & 1 & \end{matrix}\right|$ and $\left|\begin{matrix} 11 & yu & 5 \\ 1 & & 1 \end{matrix}\right|$ meaning respectively $x+5$ and $11+5$.

The abbreviation *gu* for guṇitam (multiplication) is sometimes put in but very often dropped:

*** e.g. $\left\|\begin{matrix} 2 & 5 \\ 1 & 2† \end{matrix}\right\|\begin{matrix} 3 & gu^{\circ} & 7 \\ 1 & & 2† \end{matrix}\left\|\begin{matrix} 4 & gu \cdots\cdots \\ 1 & \end{matrix}\right.$

The text is mutilated here but it is clear that *gu* is omitted between $\frac{2}{1}$ and $\frac{5}{2}$†.

* B. M. III pp. 187, 188; 67 verso and recto.

** B. M. III Folio 59 recto; p. 215.

*** B. M. III Folio 25 verso; p. 196, last line above the foot-note.

It is likely that many abbreviations are 'ad-hoc' inventions of the scribes themselves, who must belong to a period much later than the original author and one need not therefore attach to them any historical or mathematical importance.

There is one unique symbol whose use is fairly consistent but for stray exceptions evidently due to the carelessness of the scribes. The sign of the cross (†) is used to denote Riṇa or negative and placed to the right of the number which it qualifies. The symbol is peculiar to the Bakshālī Manuscript and does not occur elsewhere in Hindu mathematical works. It is also an operational symbol for subtraction. In the Nyasa for a problem on profit and loss,* we have (†) after 56 in $\left|\begin{array}{cc|c} 16 & 6 & \text{riṇam } 56\dagger \\ 5 & 1 & 1 \end{array}\right|$ to indicate that 56 is the loss. Again in another, ** we have $\left|\begin{array}{cc|cc} 1 & 2 & 3 & 6 \\ 2 & 5 & \cdot & 4\dagger \end{array}\right|$ where the significance of the symbol † is not clear. Apart from these two anomalies, wherever the cross † is used, the number preceding it is meant to be subtracted. On account of some resemblance (suggested first by Thibaut) between this cross-mark and the Diophantine mark ⋔ used for a similar purpose but in a different position, Kaye is inclined to perceive here a Greek influence; while Hoernle suggests that the Indian symbol may be an abbreviation of one of the numerous words riṇa, kanita, ūna, kshaya meaning diminution and makes a particular reference to *ka*, written exactly like the Bakshālī cross-symbol in the Asoka script. But Kaye gravely points out the danger of tracing an isolated symbol back through the ages, while he himself complacently falls into the same error with respect to the Diophantine symbol. In this connection we may recollect D. E. Smith's[4] warning that the Arithmetic of Diophantus now extant was written in the 13th century (a thousand years after the original) and we have to allow for the possible interpolations of medieval copyists. In the early medieval period in India, say from the time of Bhaskara, the dot-symbol (·) for the negative became the fashion. In a subsequent section we discuss two interesting uses of the dot-symbol in the Bakshālī Manuscript. The absence of the use of the dot-symbol for negative quantities is an important evidence that the Bakshālī work must have preceded Bhaskara.

* B. M. III p, 220. 63 recto. ** B. M. III 18 recto, p. 211.

§ 4. The Decimal Notation and the absence of Word-numerals

Throughout the Bakshālī Manuscript, some special numerals are used and the decimal notation employed in much the same way as in the modern notation. There is no attempt made to mention big numbers in words. The only other early Indian work systematically avoiding word-numerals is the[5] Aryabhatīya containing an ad-hoc notation for expressing big numbers. The fashion of word-numeration, i.e., using a word to connote the idea of a number, say 'eyes' for '2', 'tithi' (lunar days of the half month) for '15', 'teeth' for '32' and so on, was set by Varāhamihira in the sixth century and since then became an extraordinarily popular method of numeration in Hindu mathematical and other works. The systematic elimination of word-numerals in the present work is a sure index that it belongs to a period when the word-numeration had not yet become the general fashion. The employment of decimal notation in the present work is an additional evidence that the notation flourished in India even in the early centuries of the Christian era. But Kaye argues the other way about and ascribes the work to the medieval period on account of the decimal notation. Every Hindu work from the earliest times including the Vedic period shows intimate familiarity with the decimal numeration, one, ten, hundred, thousand, etc. which naturally led sooner or later unmistakably to the decimal notation with its ten figures and place-value. The play of position is a conspicuous feature of early Indian arithmetical symbolism.[2]

§ 5. The Symbol for the Unknown

In the Bakshālī Manuscript the dot (·) is used with apparently two different kinds of significance but with the same underlying idea of 'void', 'gap', or 'emptiness'. The dot, primarily a symbol of 'emptiness' must have become secondarily a symbol for the unknown or absent quantity. For, in one of the Sūtras* we are directed to put any desired number in the place of the unknown marked by the sign of 'emptiness' (Sūnya). An analogous use[3] of the zero for the unknown quantity in a proportion appears in a Latin manuscript of some lectures by Gottfried Wolack in the University of Erfont in 1467 and 1468. When the Sūnya stands for the unknown, a coefficient 1 is always set under it, as for example, $\binom{y\bar{a}}{1}$ in Bhaskara's Bijaganita. Another curious feature of the Bakshālī dot-mark is that it does duty simultaneously for several

* B. M. II. Folio 22 verso and 23 recto, pp. 122, 123.

unknowns, probably because it was felt to be more general and abstract than a literal or any other symbol that might be thought of:

$$e.\ g. \left|\begin{array}{cccccc} \cdot & 5 & yu & mu & \cdot \\ 1 & 1 & & & 1 \end{array}\right| \begin{array}{cccccc} s\bar{a} & \cdot & 7\dot{t} & mu & \cdot \\ & 1 & 1 & & 1 \end{array}\Big| .$$

which means $\sqrt{x+5} = y, \quad \sqrt{x-7} = z.$

Kaye believes that the ambiguous use of the dot-symbol is an indication of the lack of an efficient symbolism and is perplexed by the denominator unity for the dot-symbol. He fails to appreciate that the denominator unity serves to indicate that the dot-symbol is a symbol for the unknown and not the symbol for 'zero'. Here we have again another hint that the contents of the manuscript must belong to a period when the literal notation for the unknown did not come into general use in Hindu works.

The symbol for 'zero', the negative sign, and the symbol for the unknown seem to have a common ancestry and their nebulous beginnings are perhaps reflected in the Bakshālī Manuscript.

§ 6. The Rule of Regula Falsi

David Eugene Smith remarks[4] 'Awkward as this seems, the rule (explained below) was used for many centuries, a witness to the need for and value of a good symbolism'. Kaye believes in this dictum. The rule of double false to solve $ax+b=0$, gives

$$x = \frac{f_1 g_2 - f_2 g_1}{f_1 - f_2} \text{ where } ag_1 + b = f_1 \text{ and } ag_2 + b = f_2, \quad f_1, f_2$$

being called the two false values or failures corresponding to g_1 and g_2 respectively. In effect, I should say that this method is the same as that in analytical geometry to find the co-ordinates of the point where the line joining (g_1, f_1) and (g_2, f_2) meets the x-axis. From another point of view, the method may be deemed to involve the principle of linear interpolation. After all, does not the theory of interpolation involve the introduction of not two, but several falses hovering about the true value one is in quest of? When the truth happens to lie deep hidden, we are sometimes forced to accept the nearest falsehood as truth. Hence the method of the double false is not an awkward one but envisages an essential process of making falses themselves yield truth, a precursor of modern interpolation theory.

But the above rule ought not to be confused with the Indian Ishṭakarma which is an operation with an assumed number, used in

cases where the final result arrived at after a series of manipulations is always proportional to the number originally assumed. The required number can be obtained by Trairāśika or proportion corresponding to the given final result or Driṣya. Algebraically, the Ishtakarma is equivalent to solving the simple equation $ax=b$ and arriving at $x=\frac{bm}{am}$, where am is the value of the left-hand side when $x=m$ (Ishṭa). It is important to note that Ishtakarma is a useful device to evaluate x when a is a complicated expression, say $(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{4})$. When a is a straightforward simple coefficient, the Hindu mathematician would straightaway utilise his Rule of Inverse Operations and get the value of x. The Ishtakarma must be regarded therefore more as an appendix to Trairāśika (proportion) than as a degenerate case of 'Regula Falsi'.

The use of Regula Falsi in the Bakshālī Manuscript as conceived by Kaye† is not the same as the one used in the middle ages but a fanciful deduction based on a misunderstanding of an ingenious method of generalised Ishṭakarma to be presently explained. The author of Bakshālī Manuscript solves the equations

$$x_1+x_2=16,\ x_2+x_3=17,\ x_3+x_4=18,\ x_4+x_5=19,\ x_5+x_1=20$$

by assuming a tentative value (Ishṭa), say 7 for x_1 and derives from it successively the values, 9, 8, 10, 9 for x_2, x_3, x_4, x_5, and thence $x_5+x_1=16$. But from the structure of the equations, it is noticed that any decrease or increase in x_1 involves a corresponding and same decrease or increase in x_3 and x_5, so that the decrease or increase gets doubled for x_1+x_5. Hence the true value for x_1 can be obtained from the tentative value by decreasing or increasing it by *half* the total deficit or excess in the actual given value of x_1+x_5 as compared with the value based on our assumption. Thus the assumed value 7 should be increased by 2 and $x_1=9$. Thence $x_2=7$, $x_3=10$, $x_4=8$, $x_5=11$.

This method again is peculiar to Bakshālī Mathematics and I have not come across the like of it elsewhere. To trace this to the medieval Regula Falsi and thence infer an evidence that the work belongs to medieval times is a far-fetched argument.

The Hindu mathematician has never used the medieval 'Regula Falsi' to solve a simple equation of the type $ax+b=cx+d$. Kaye ought to have been aware of [5] Aryabhata's solution of this in the

† Vide B. M. I pp. 32, 33.

form $(d-b)/(a-c)$, as well as similar solutions presented in the text of Bakshālī Manuscript† and yet asserts that the Regula Falsi was used by the early Hindu mathematicians on account of lack of efficient symbolism. Regula Falsi as interpreted by Kaye, at least with reference to the Bakshālī Manuscript is regular falsehood.

§ 7. The Square-root rule and the Process of Reconciliation

Another striking feature in the mathematical contents of the Bakshālī Manuscript is the chapter on the square-root rule, where we have

$$\sqrt{a^2+\gamma} = a + \frac{\gamma}{2a} - \frac{(\gamma/2a)^2}{2\,(a+\gamma/2a)}$$

an approximation found in exactly the same form also in an Arab work of the twelfth century. This has given rise to a speculation that the present work may therefore be later than the twelfth century. But it must be noted that the rule is merely a corollary of the general square-root rule found explicitly in all Indian mathematical works, at any rate from Aryabhata onwards. The earliest evidence of a concrete numerical application of this rule even to a higher order than the one indicated by the above formula is to be found in the Sulva Sūtras, which contain the approximation $1+\frac{1}{3}+\frac{1}{3\cdot4}-\frac{1}{3\cdot4\cdot34}$ for $\sqrt{2}$. This formula can be easily explained by the usual Indian algorithm for finding square-roots, (the same as the one adopted even to-day):

$$\begin{array}{r|l}
\begin{matrix}2\\ 1\end{matrix} & 1+\frac{1}{3}+\frac{1}{3\cdot4}-\frac{1}{3\cdot4\cdot34} \\
\hline
2+\frac{1}{3} & \begin{matrix}1\\ 7\ 9\\ \hline 2/9\end{matrix} \\
2+\frac{2}{3}+\frac{1}{3\cdot4} & 2/9+\frac{1}{3\cdot4\cdot3\cdot4} \\
 & \overline{\qquad\qquad\qquad} \\
2+\frac{2}{3}+\frac{1}{3\cdot2}-\frac{1}{3\cdot4\cdot34} & \quad -\frac{1}{3\cdot4\cdot3\cdot4} \\
 & \quad -\frac{1}{3\cdot4\cdot3\cdot4}+\frac{1}{3\cdot4\cdot34\cdot3\cdot4\cdot34} \\
\hline
\end{array}$$

and so on, ad libitum.

Further, it is important to note that the Bakshālī rule ought not to be so narrowly interpreted that it gives only the second approximation but is suggestive of a process which permits repeated application to

† B. M. III p. 173 (9 verso).

any desired order of approximation. Brahmagupta[8] mentions a similar rule in connection with the square-root of a number in sexagesimal notation.

The occasion for finding the square-root approximation is a very fanciful one. For, it is used to find the number of terms 't' of an arithmetical progression, when the sum s, the first term a and the common difference d are given. t is given by the formula:

$$t = \frac{d - 2a + \sqrt{(2a-d)^2 + 8ds}}{2d}$$

while $$s = t\left(a + \overline{t-1}.\frac{d}{2}\right)$$

Both these occur in all early Hindu mathematical works. The new feature in the present work is the following:

If an approximation to $\sqrt{(2a-d)^2 + 8ds}$ be q_1 and the corresponding value of t be t_1 and the corresponding sum s_1, the verification of the approximation consists in noting that

$$s = s_1 - \frac{e_1}{8d} \text{ where } e_1 = q_1^2 - \{(2a-d)^2 + 8ds\}$$

The proof of this result is extremely simple, but Kaye puts up a long rigmarole.

In fact, if $s_1 = t_1\left(a + \overline{t_1 - 1}.\frac{d}{2}\right)$, it follows immediately

that $$t_1 = \frac{d - 2a + \sqrt{(2a-d)^2 + 8ds_1}}{2d},$$

and hence $$q_1^2 = (2a-d)^2 + 8ds_1$$
$$= (2a-d)^2 + 8ds + 8d\,(s_1 - s);$$

$$\therefore \quad e_1 = 8d\ (s_1 - s) \text{ or } s = s_1 - e_1/8d.$$

This method of verification is obviously applicable whatever be the degree of approximation. It is nothing peculiarly adapted to the approximation of the second degree. But the value of e_1 is either $\left(\frac{\gamma}{2a}\right)^2$ or $\frac{(\gamma/2a)^4}{4(a+\gamma/2a)^2}$ according as the first or second approximation is taken.

The author of the Bakshālī manuscript seems to take a peculiar pleasure in making huge calculations to verify

$$e_1 = 8d(s_1 - s)$$

in a number of examples. Mathematically, all this work is futile and meaningless, since the summation of an Arithmetical Progression to a non-integral number of terms is not practical mathematics. But such a summation seems to be very common in the days of Chaturveda, Sridhara and Mahavira, i.e. roughly the ninth and tenth centuries. Mahavira triumphantly asks "Give out the first term and the common difference respectively in relation to two series, in which $\frac{31}{150}$ is the sum, while $\frac{3}{4}$ in one case is the common difference, and $\frac{4}{5}$ the number of terms and in the other case $\frac{1}{3}$ the first term and $\frac{4}{5}$ the number of terms."(7) Very likely the Bakshali arithmetician who revels in summation of series to an irrational number of terms belongs to the same school and is perhaps contemporaneous with the writers just quoted. A typical Bakshālī problem involving square-root approximation may be given by way of illustration: 'A certain person goes 5 yojanas on the first day, and 3 yojanas more on each succeeding day. Another who travels 7 yojanas per day has a start of 5 days. When will they meet, say O! the best of mathematicians.'† The answer given in the text is $\frac{7+\sqrt{889}}{6}$ days or $6\frac{4}{29}$ days approximately, i.e. 6 days, 8 ghatikas,* 16 vighatis, $33'''$, $6''''\frac{6}{25}$. The futility of the result transcends one's imagination. But Dr. Bibhutibhushan Datta(2) comments:

"The expression for the time in which two persons will meet contains a surd quantity. So the two persons will never meet"!! A worthy comment! The problem is quite sensible but not the above solution. The mathematician has strayed hopelessly away from common sense, carried away by his unqualified faith in the algebraic formula. The correct answer is 6 days, 7 ghatikas and 30 vighatis, quite rational and free from surds. The reader can easily verify this result.

It takes perhaps centuries to outgrow such impossible fantasies as those exhibited in the problems on square-root approximations. We are glad to find Bhaskara of the twelfth century entirely free from them. He was, perhaps, the earliest Hindu mathematician to perceive the beauty and importance of integers and integral solutions. His marvellous chakravāla method, whose import is even to this day imperfectly understood, of solving $x^2 - Ny^2 = 1$ (N a non-square positive integer) in positive integers is a remarkable instance of 'the sense' for the integer. The Bakshālī mathematician is too ancient to have cultivated this 'sense.'

† B. M. III p. 178. 6 recto.

* 1 Ghatika $= \frac{1}{60}$ day $=$ 60 vighatsi.

§ 8. Series and Sequences

The problems on series and sequences are very original, varied and interesting. Kaye, however, holds a different opinion and gives a very confused and misleading analysis of the types of problems. In most of the problems, the sum to n terms happens to be proportional to the first term and therefore Ishtakarma can apply. The several types are indicated below in modern notation with T_n and S_n denoting respectively the n^{th} term and the sum to n terms. We discuss *eight* types according to the character of T_n.

(1) *Arithmetical Progression.*

$T_n = nT_1 \pm (a + d \cdot \overline{n-1})$†. This can be reduced to

$T_n \pm d = n(T_1 \pm d) \pm a$, or changing the notation

$V_n = nV_1 + a$, which is an arithmetic progression with common difference V_1 and first term $V_1 + a$.

(2) *Geometrical Progression.*

$$T_n = ar^{n-1} + ab\,\frac{r^{n-1}-1}{r-1} \quad ††$$

$$= a\left\{\frac{b+r-1}{r-1} \cdot r^{n-1} - \frac{b}{r-1}\right\}$$ which is derived from a G.P. by increasing every term by a constant.

$$\therefore \quad S_n = a\left\{\frac{b+r-1}{(r-1)^2} \cdot (r^n - 1) - \frac{nb}{r-1}\right\}$$ which is proportional to the first term a if b, r, n be given.

Illustration: $S_n = 329$, $r = 3$, $n = 5$, $b = \frac{3}{4}$. ††

Hence $a = 2$, obtained in the text by Ishtakarma assuming $a = 1$ and setting the corresponding S_n (from which the true value of a comes out by proportion).

(3) $T_n = a\ S_{n-1}$.*

Hence $S_n = S_{n-1} + T_n = S_{n-1}(1+a) = S_{n-2}(1+a)^2 = \cdots\cdots$
$= S_1(1+a)^{n-1}$ or $T_1(1+a)^{n-1}$

(4) $T_n = n \cdot S_{n-1}$**.

This is the most interesting of all.

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† B. M. II P. 122. 22 verso.

†† B. M. II P. 143, 51 verso; B. M. III p. 164.

* B. M. III P. 236 (42, verso) where $S_n = 54$, $a = \frac{1}{2}$, $n = 3$. Hence $T_1 = 24$.

** B. M. III P. 194 (23, verso) where $S_n = 300$, $n = 4$.
Hence $T_1 = 5$, $T_2 = 10$, $T_3 = 45$, $T_4 = 240$,

Since $T_n = S_n - S_{n-1} = S_n \cdot \frac{n}{n+1}$, $S_{n-1} = \frac{S_n}{n+1}$.

Similarly $T_{n-1} = S_{n-1} \cdot \frac{n-1}{n} = \frac{n-1}{n(n+1)} S_n$ and $S_{n-2} = \frac{S_n}{(n+1)n}$, $\cdots$

$$T_{n-r} = \frac{(n-r)\, S_n}{(n+1)n \cdots (n-r+1)},$$

$$S_{n-r-1} = \frac{S_n}{(n+1)n \cdots (n-r+1)} \quad (0<r<n-1)$$

$$T_2 = \frac{2S_n}{(n+1)n \cdots 3} = 2T_1.$$

Hence $T_1 = \frac{2S_n}{(n+1)!}$ and $S_n = \frac{1}{2} T_1 \cdot (n+1)!$

Thus, given the law of formation of the sequence, and the sum to n terms, and the number of terms n, the series can be determined. Here again the sum is proportional to the first term and 'Iṣhṭa-karma' operation is applicable. This is the method adopted in the text which is indeed preferable in numerical work to the algebraic expressions that we have given.

(5) $T_n = nS_{n-1} \pm (a + n - 1 \cdot d) = nS_{n-1} + U_n$ (say).

$\therefore \quad S_n = (n+1)\, S_{n-1} + U_n$

$S_{n-1} = nS_{n-2} + U_{n-1}$

$\cdots\cdots\cdots\cdots\cdots$

$S_2 = 3S_1 + U_2 = 3T_1 + U_2$

Hence, eliminating $S_{n-1}, S_{n-2}, \cdots S_2$, we have

$$\begin{aligned} S_n &= U_n + (n+1)\, U_{n-1} + (n+1) \cdot nU_{n-2} + \cdots + (n+1)\, n\, (n-1) \cdots 4U_2 \\ &\quad + 3 \cdot (n+1)\, n\, (n-1) \cdots 4 \cdot T_1 \\ &= \tfrac{1}{2} T_1 \cdot (n+1)! + U_n + (n+1)\, U_{n-1} + \cdots \\ &\qquad + (n+1)\, n\, (n-1) \cdots 4U_2 \end{aligned}$$

The example in the text † is somewhat like this:

A has something with $1\frac{1}{2}$, B twice as much as A with $2\frac{1}{2}$, C three times as much as A and B and $3\frac{1}{2}$ in addition, D four times as much as A, B, and C and $4\frac{1}{2}$ in addition, their total possessions are 222. What is the possession of A?

This is interpreted by Kaye symbolically as:

$A = x\ (1+1\frac{1}{2})$, $B = 2A + 2\frac{1}{2}x$, $C = 3A + 3B + 3\frac{1}{2}x$, $D = 4A + 4B + 4C + 4\frac{1}{2}x$, and $A + B + C + D = 222$.

† B. M. II 24 verso p. 124.

I believe this interpretation is incorrect though it may justify the solution of the text and the use of Ishṭakarma. A proper interpretation would be

$$A = x + 1\tfrac{1}{2},\ B = 2A + 2\tfrac{1}{2},\ C = 3A + 3B + 3\tfrac{1}{2},\ D = 4A + 4B + 4C + 4\tfrac{1}{2}$$

$$A + B + C + D = 222.$$

Though $x = 1$ satisfies by chance this set of equations, the set of equations is not strictly amenable to Ishṭakarma by mere proportion. A different kind of Ishṭakarma similar to the one described on p. 7 above should have been adopted here with due care. The Bakshali mathematician has not done this and therefore has erred in his method. Such an error has been repeated in three similar problems †† and we get a glimpse of the limitations of the Ganakaraja.

(6) $T_n = nT_{n-1}$

Here, $T_n = n!$ if $T_1 = 1$ and there is no compendious formula for the exact summation of such terms.

(7) $T_n = nT_{n-1} \pm (a + \overline{n-1} \cdot d)$

This can be reduced to the form $U_n = nU_{n-1} \pm a$, which also does not admit of summation by any formula.

(8) $T_n = (1 - a_1)(1 - a_2) \cdots (1 - a_{n-1})a_n.$

We are reminded of Euler's identity:

$$(1 - a_1) + a_1(1 - a_2) + a_1 a_2(1 - a_3) + \cdots + a_1 a_2 \cdots a_{n-1}(1 - a_n)$$
$$= 1 - a_1 a_2 \cdots a_n.$$

Changing a_r to $1 - a_r$, we have the Indian counterpart of Euler's identity, viz.

$$a_1 + a_2(1 - a_1) + a_3(1 - a_1)(1 - a_2) + \cdots + a_n(1 - a_1)(1 - a_2) \cdots (1 - a_{n-1})$$
$$= 1 - (1 - a_1)(1 - a_2) \cdots (1 - a_n).$$

Problems involving this identity are common in Hindu mathematical works from at least the 8th century onwards. No wonder that the Arithmetical Papyrus of Akhim (9th century?) contains such problems. But Kaye would prefer to make Akhim the inventor of the above identity and hence infer that the Bakshālī Manuscript must belong to the tenth century or later.

†† B, M, II 24 recto, 25 recto and verso, pp. 123-125.

§ 9. General Remarks

Having pointed out the famous peculiarities of the Bakshālī Manuscript in respect of its mathematical content, exposition and symbolism we will conclude with a few general remarks on the age and style of the work. Peculiarities apart, the Bakshālī text is more or less a replica of other Hindu mathematical works, such as the Ganitasara Sangraha. It contains in common with them the following:

(1) Practical and commercial problems such as the computation of fineness of gold, (2) problems on income and expenditure, (3) motion-problems, (4) profit and loss, (5) interest, (6) bills of exchange or hundika, and (7) miscellaneous problems involving literary and social references.

In addition, we may note that the way in which the solutions are given in the Bakshālī Manuscript in such a general form as to be nearly algebraic in character even though no adequate symbolism is employed, is just characteristic of all Indian works, since the day when Aryabhata set the fashion in this respect. There are, however, certain omissions, for example, expressions for the sums of squares and cubes, the indeterminate equations of the first and second degree, shadow problems, permutation and combination. These may be either apparent or real, because our manuscript is mutilated and several leaves are stuck together. In the latter case, the omission is significant and bespeaks an early date of composition, i.e. prior to the tenth century by which period these problems had become well known at least in Indian works. Among further evidences of early composition, we have already mentioned the meagre elaboration of problems dealt with in greater detail in the works of Brahmagupta Mahaviracharya and Bhaskara, the studied absence of the word-numeral notation, the use of the cross-symbol † instead of the dot ⊙ to denote a negative quantity, the use of the dot ⊙ to denote an unknown quantity, and a certain careless application of the Ishṭa-karma in as many as five problems.

The employment of the modern place-value arithmetical notation is regarded by Kaye as indicating a late period, as also what he believes to be non-Indian material, viz. the negative sign resembling the Diophantine symbol, the use of the Regula-falsi, the square-root approximation, the use of the Sexagesimal notation for example in the conversion of days into ghatikas, vighatis, and so forth, and the use of the terms dinara, dramma, satera and so forth. How much the

non-Indian material in Bakshālī Manuscript is really indigenous has been sufficiently brought out in the preceding pages.

However late we may wish to place the manuscript, we cannot well go beyond the tenth century. We therefore come near the times of Sridhara, Mahavira, and Chaturveda with whose works the Bakshālī work has many points in common. Evidences of language, script and such special terms as Hundika do not contradict this view. Kaye's insistence of the twelfth century as the probable date of composition is wholly untenable.

The systematic presentation of the working steps and methods of verification, along with the use of a certain kind of syncopated notation are of great pedagogic interest, quite in keeping with the fact that the work was written for the express benefit of young boys†. The simple and diffuse style adopted suits this aim.

The Bakshālī Manuscript may lack the subtlety of the Aryabhatiya; it may not possess the profundity of the Brahmasphuta Siddhanta; it may be a poor literary specimen by the side of the rich poetry of Ganitasarasangraha; but no one can deny its great value for a practical teacher. If we recollect that Bakshālī was at a distance of only 70 miles from Taxila, one of the renowned University centres in Ancient India, we have here perhaps a glimpse of the lecture-notes of a University professor of bygone days. To do the Ganakaraja justice, we may add that flashes of genius gleam forth here and there in the method of reconciliation of square-root approximations, in dreaming of series and sequences governed by elegant and sometimes complicated laws, in the solution of systems of linear equations by suitable change of variables and assumption of tentative solutions, and lastly in the construction of problems of humorous and mythological interest.

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2. The Bakshālī Mathematics by Bibhutibhusan Datta, *Bulletin of the Cal. Math. Soc.* XXI, 1929.

† B. M. II p. 142, 50r. The Colophon reads:
. . . . Vasishta putraha.
Sikasyartheputra pautra upayogam bhavatu likhitam chchhajaka putra ganaka rāja brahmaṇena . . .

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TOPICS IN THE THEORY OF ANALYTIC FUNCTIONS (II):*

(Singularities, Over-convergence and Gaps; Harmonic Functions; Maximum-modulus principle; Normal Families of Analytic Functions.)

BY

V. GANAPATHY IYER

1. Singularities of analytic Functions. An analytic Function (according to Weierstrass, I, § 7 - 10) is entirely determined from any one of its elements. Therefore, theoretically, the singularities of an analytic function must be completely determinable if we are given a single element of the function. But the practical determination of the position and nature of the singularities of a function starting from one of its elements is extremely difficult. We have seen that there is at least one singular point on the circumference of the circle of convergence of a power series. Since an analytic function is merely the totality of its elements it follows that a singular point of an analytic function is either (i) situated on the circumference of the circles of convergence of its elements or (ii) a limit point of the singularities of type (i). So it is sufficient to study the nature and position of the singularities on the circumference of the circles of convergence of a power series.

* The first article appeared in *The Mathematics Student*, Vol. 5, (1938), pp. 1 - 24. This will be referred to as I in the sequel. The notations in I are used throughout.

1·1. *The general test for a singularity*†.

Let $P_a(z)$ be the power series

$$\sum_{n=0}^{\infty} a_n (z-a)^n \qquad \dots \quad (1)$$

Let z_0 be a point on the circumference c (P_a) of the circle of convergence C (P_a) of the series. *Then z_0 is a singular point if and only if the radius of convergence of the immediate continuation P_b (z) of P_a (z) for any point b on the radius (a, z_0) is exactly equal to $|b-z_0|$.* The condition we arrive at by using the above criterion is, in general, unwieldy and so is not useful in practice. Simplifications* of the test have been introduced and with their aid several interesting results on the position of singularities have been obtained*. We shall confine ourselves to mentioning a few of these.

1·2. In stating the results we shall take $a=0$ in P_a (z) so that we get

$$P_0 (z) = \sum_{n=0}^{\infty} a_n z^n \qquad \dots \quad (2)$$

If $a_n > 0$, or more generally, $|\text{amp.} (a_n)| \leqslant \alpha < \pi/2$ *then $z=\rho$ on $c(P_0)$ is a singular point where ρ is the radius of convergence.* Again, let $[\lambda_n]$, $0 \leqslant \lambda_1 < \lambda_2 \cdots < \lambda_n \cdots$ denote the indices of (2) for which $a_{\lambda_n} \neq 0$, that is, the coefficients are non-vanishing. Then, if $\lambda_n/n \to \infty$ *as* $n \to \infty$, *every point on c (P_0) is a singular point. In particular this is true if* (*i*) $\lambda_{n+1} - \lambda_n \to \infty$ *and* (*ii*) $\lim \lambda_{n+1}/\lambda_n > 1$. In this case the analytic function cannot be continued beyond the circle C (P_0) which is therefore its domain of existence. The function is uniform in C (P_0) and the circumference c (P_0) is called the natural cut or boundary for the function. In this connection we wish to mention the following curious result. Let us start with *any* series $\sum a_n z^n$ and let us consider the totality of the series of the form $\sum \pm a_n z^n$ obtained by distributing in all possible manner the signs + and − among the coefficients (a_n) of the given series. Then we can find at least one among these series for which c (P_0) is the natural boundary. Indeed, the cardinal number of such series is the same as that of the continuum.**

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† See P. Dienes, *The Taylor Series*, Oxford (1931), the chapter on singularities.

* See P. Dienes, last cited. Also, S. Mandelbrojt, "Modern Researches on the Singularities of functions defined by Taylor's Series", *The Rice Institute Pamphlet*, Vol. 14 (1927), No 4.

** For proofs of these and further results see P. Dienes and Mandelbrojt last cited.

2. Uniform convergence in a domain. Let $[f_n(z)]$ be a sequence of functions each of which is regular in a domain D. *If the sequence converges uniformly on any closed set interior to D then the sequence is said to converge uniformly in D.* In this case the limit function is also regular in D; here we suppose that a constant (finite or infinite) is regular in D, the limit of the sequence being an infinite constant when $|f_n(z)| \to \infty$ uniformly on any closed set interior to D. In this connection we wish to mention Weierstrass's theorem on the uniform convergence of a sequence of functions. Let D be a domain such as contemplated in Cauchy's formula [(10), § 3·6, p·8, I]. Let $[f_n(z)]$ be a sequence of functions each of which is regular in D and continuous on the boundary B (D). Then the uniform convergence of $[f_n(z)]$ on B (D) implies its uniform convergence in D. Moreover, the sequence $\left[f_n^{(k)}(z)\right]$ of the k^{th} derivatives also converges uniformly in D to $f^{(k)}(z)$ where $f(z)$ (here supposed to be finite) is the limit of the original sequence.

2·1. Over-convergence. Let us consider a sequence $[f_n(z)]$ of functions converging uniformly in a domain D and let us further suppose that the sequence $[f_n(z)]$ does *not* converge uniformly in any *larger* domain containing D. If this be the case we might call D a *complete domain** of uniform convergence of the sequence $[f_n(z)]$. Now although the whole sequence $[f_n(z)]$ does not converge uniformly in a larger domain D_1 containing D it *can happen* that a *sub-sequence* $[f_{n_k}(z)]$ of $[f_n(z)]$ *converges uniformly in such a domain* D_1. When this is the case *the original sequence is said to over-converge outside D.* In this case the limit function $f(z)$ of $[f_n(z)]$ is necessarily regular in the larger domain D_1 and the sub-sequence converges to $f(z)$ in D_1 as well. In order that a sequence over-converges outside D it is necessary and sufficient that a sub-sequence of $[f_n(z)]$ should converge uniformly in the neighbourhood of a point of B(D).

2·2. The phenomenon of over-convergence explained in the previous paragraph has been mostly studied in the case when $f_n(z) = S_n(z) = \sum_{k=0}^{n} a_k z^k$, the sum to first $(n+1)$ terms of the power series $P_0(z)$. By the well-known properties of a power-series, $[S_n(z)]$ cannot converge at any point outside $C(P_0)$. So the interior of $C(P_0)$ is a complete domain of uniform convergence of $[S_n(z)]$.

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* A sequence can converge uniformly to different functions in different non-overlapping domains. So there can be more than one complete domain of uniform convergence, see I, § 6.

The main result in connection with the over-convergence of the sections $[S_n(z)]$ of a power-series $P_0(z)$ is due to A. Ostrowski *. *Let* $P_0(z)$ *be such that there are an infinity of pairs of integers* (p_k, q_k), $k=1, 2, \ldots$ *such that* (*i*) $a_n=0$ *for* $p_k < n < q_k$ *and* (*ii*) $\lim\limits_{k\to\infty} \frac{q_k}{p_k} > 1$. *Then the sub-sequence* $[S_{p_k}(z)]$ *of* $[S_n(z)]$ *converges uniformly in the neighbourhood of any regular point* z_0 *of the function* $f(z)$ *defined by* $P_0(z)$, z_0 *being on* $c(P_0)$. That is, under these conditions $[S_n(z)]$ over-converges outside $C(P_0)$.

2·3. **Gaps.** In the above result on over-convergence we come across *an infinity* of *gaps* in the coefficients $(a_0, a_1, a_2, \ldots)$ of the power series $P_0(z)$. These gaps (p_k, q_k) are such that the width $q_k - p_k$ of the gaps are not less than dp_k ($d>0$ fixed) for all $k>$ some k_0. When a power series $P_0(z)$ possesses such gaps we shall call it a *gap-series* (*seriés lacunaires*). The hypothesis on the existence of gaps in connection with the theorem on over-converges is not accidental. It can be proved that if $[S_n(z)]$ over-converges then

$$P_0(z) = Q_0(z) + R_0(z) \qquad \ldots \quad (3)$$

where $Q_0(z)$ is a gap-series and $R_0(z)$ has a greater radius of convergence than $P_0(z)$. The over-convergence theorem in § 2·2 easily leads to the case (ii) of the second result on singularities mentioned in § 1·2.†

3. **Harmonic functions.** A real function $u(z) = u(x, y)$, $z = x + iy$. is said to be harmonic in a domain D when it is the real part of a function $f(z) = u(z) + iv(z)$ regular in D. Since $i f(z)$ is also regular in D, the complex or the imaginary part $v(z)$ is also harmonic in D. From (6) of I (p. 6) we deduce that any harmonic function $u(z)$ is a solution of the Laplace's equation

$$\Delta U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0. \qquad \ldots \quad (4)$$

Conversely any continuous solution of (4) is a harmonic function.

3·1. Let $z = x + iy$, $x = r\cos\theta$, $y = r\sin\theta$ and let $u(z) = u(x, y) = u(r, \theta)$ be a function harmonic in a domain D. Let z_0 be any point of D

* See P. Dienes, last cited, the chapter on Over-convergence and Gap theorems See also E. C. Titchmarsh, *Theory of Functions*, pp. 220-224.

† For proofs of these results see P. Dienes, last cited. Also, see Bourion *L'ultraconvergence dans les series de Taylor*; Actualites Scientifiques No. 472.

and let C (z_0, r) be a circle in D. Then one of the most important properties of harmonic functions is given by the relation

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(r, \theta)\, d\theta \qquad \dots \quad (5)$$

That is, *the mean value of $u(z)$ along the circumference of the circle $C(z_0, r)$ is equal to its value at the centre.* This result is easily proved by integrating a power series along the circumference of a circle and equating the real parts. Conversely *any function $u(z)$ continuous in a domain D satisfying the relation (5) for any circle inside D is harmonic in D.*

3·2. **Sub-harmonic and Super-harmonic functions.** Let $u(z)$ be a real function continuous in a domain D. If *for every circle $C(z_0, r)$* inside D the relation

$$u(z_0) \leqslant \frac{1}{2\pi}\int_0^{2\pi} u(r, \theta)\, d\theta \qquad \dots \quad (6)$$

holds, the function $u(z)$ is said to be *sub-harmonic* in D. If, on the other hand, the relation

$$u(z_0) \geqslant \frac{1}{2\pi}\int_0^{2\pi} u(r, \theta)\, d\theta, \qquad \dots \quad (7)$$

holds, $u(z)$ is said to be *super-harmonic* in D. If $u(z)$ is sub-harmonic in D, then $-u(z)$ is super-harmonic in D. *A harmonic function is at the same time sub-harmonic and super-harmonic.*

3·3. One of the chief properties of a sub-harmonic function $u(z)$ defined in a domain D is that *$u(z)$ cannot attain its upper bound (or maximum value) at a point of D unless $u(z)$ is a constant in D.* This result is easily proved from relation (6). If $u(z)$ is continuous on the boundary B (D) this upper bound will be attained at a point of B (D). Similarly *a super-harmonic function cannot attain its lower-bound (or minimum value) at a point of* D. Therefore *a function $u(z)$ harmonic in D cannot attain its upper or lower bound (maximum or minimum value) at a point of D unless it is a constant.* From this we deduce that *if $u_1(z)$ and $u_2(z)$ are harmonic in D and continuous on B (D) then the equality $u_1(z) = u_2(z)$ at points of B (D) implies $u_1(z) \equiv u_2(z)$ throughout D + B (D).*

3·4. **Dirichlet's problem.** The most extensively studied subject in connection with the theory of harmonic functions is what is known as Dirichlet's problem. The last statement in § 3·3 above shows that

if u (z) is harmonic in a domain D and continuous on B (D) then there cannot exist another function harmonic in D taking the same set of continuous boundary values on B (D). Dirichlet's problem concerns itself with the solution of the problem arising from the converse of this proposition. Let v (z) be a *given* function continuous on the boundary B (D) of a domain D. Is it possible to find a function u (z) harmonic in D such that the boundary value of u (z) on B (D) is just v (z)? With suitable restrictions on the nature of the boundary B (D) of the domain D the above problem has been completely solved. References are made elsewhere* for a detailed account of these and further properties of harmonic functions.

4. **The maximum-modulus principle for analytic functions.** Let f (z) be a function regular in a domain D. Then $|f(z)|$ *cannot attain its upper bound (or maximum) at a point of D unless f (z) is a constant.* Though this result can be proved directly it is a special case of the results of § 3·3 since it can be shown that $|f(z)|$ is sub-harmonic in D. The above result is known as the maximum-modulus principle for analytic functions. If $|f(z)|$ is continuous on B (D) then the upper bound is attained at a point of B (D). If $f(z) \neq 0$ in D, then $\dfrac{1}{f(z)}$ is regular in D and then we can also conclude that $|f(z)|$ cannot attain its minimum at a point of D.

4·1. The maximum-modulus principle has extensive applications in the theory of analytic functions. It can be extended to mutli-form functions if $|f(z)|$ is single-valued in the domain considered. As simple applications of this principle we can deduce that if f (z) be regular in a circle $|z| < R$ and $M(r, f) = \max_{|z| \leqslant r} |f(z)|$, $0 \leqslant r < R$, then $M(r, f)$ is an increasing function r and $\log M(r, f)$ is a convex function† of $\log r$. The last statement includes what is known as Hadamard's three circles theorem. Numerous other interesting results could be deduced from this principle.‡

5. **Normal families of analytic functions.** A class or a family $\{f(z)\}$ of functions each defined and regular in a domain D is said to be *normal in D when any sequence $[f_n(z)]$ of functions selected from the family contains a sub-sequence converging uniformly in D* (§ 2 above).

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* See E. Goursat, *Cours d' Analyse Mathematique*, Tome III.

† A function $\phi(x)$ is said to be convex in x for $a \leqslant x \leqslant b$ if

$$\phi(\lambda_1 x_1 + \lambda_2 x_2) \leqslant \lambda_1 \phi(x_1) + \lambda_2 \phi(x_2)$$

where x_1 and x_2 are any two points in (a, b) and λ_1 and λ_2 are numbers such that $\lambda_1 + \lambda_2 = 1$

‡ See P. Dienes, and E. C. Tithmarsh, last cited.

The idea here closely resembles the well-known result that *any infinite set* (*or class*) of numbers (real or complex) has *at least one limit point*, that is, any sequence selected from the set contains a sub-sequence converging to a limit (finite or infinite). Often, in higher analysis,* we have to consider classes of elements for which the notions of convergence and limit could be defined. But it is *not necessary* that any infinite set of these elements should possess the property mentioned above for numbers. In such cases it becomes important to consider classes of such elements which possess this special property. This is the idea behind the notion of a normal family. If we consider a class or family of functions regular in a domain D and if "*ordinary*" convergence (in the case of numbers) be replaced by "*uniform convergence in D*", then the family is normal when it has the property specified above for numbers.

5·1. We shall say that a family $\{f(z)\}$ of functions regular in a domain D is *essentially bounded* in D if every point z_0 of D is the centre of a circle $C(z_0, r)$ lying in D such that

$$|f(z)| < M, \text{ for } z \text{ in } C(z_0, r). \qquad \dots \quad (8)$$

and for each function $f(z)$ of the family; here M is a finite number which may vary with z_0 and r but is independent of the particular function $f(z)$ of the family. The most important test for a normal family can be stated as follows. *A family $\{f(z)\}$ of functions regular in D is normal in D if the family is essentially bounded in D.*

5·2. **Exceptional values and normal families.** There is an intimate connection between the values which the functions of a family omit to take and the property of the family being normal. *Let each function $f(z)$ of a family $\{f(z)\}$ of functions regular in a domain D omit to take two specified values α and β, $\alpha \neq \beta$. Then the family is normal in D*†

5·3. **Uniform convergence and normal families.** It would take us too far to consider exhaustively the results and applications of the theory of normal families. We shall content ourselves with a few general remarks. Firstly, it is possible to considerably generalise the theorem of Weierstrass on uniform convergence (§ 2) when we consider normal families. *Let $[f_n(z)]$ be normal in a domain D. If $[f_n(z)]$ converges at a set of points in D having at least one limit point in D, then the sequence converges uniformly in D.*

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* See, for instance, S. Banach, *Operations Lineaires*, Chaps. 1—4.

† The classical work in connection with the theory of normal families is P. Montel's "*Leçons sur les familles normales des fonctions analytiques*" (Borel Tracts).

5·4. **The limit functions of a normal family.** We have pointed out in § 2 that when a sequence of functions regular in D converge uniformly in a domain the limit function is either regular (in the ordinary sense in D) or is a constant finite or infinite*. In connection with normal families we can agree to regard a finite or infinite constant as regular in any domain D. *Then we can state that the class of limit functions of the various sub-sequences selected from a normal family consists of regular functions.*

5·5. Let $\{f(z)\}$ be a family of functions *regular and normal* in a domain D. *Then if the family be bounded at a single point* z_0 *of D, then the family is essentially bounded in D. The same conclusion is true if the class of limit functions does not include the infinite constant.* Again, *if a family* $\{f(z)\}$ *is normal in D, the number of roots of* $f(z)-a=0$ *in any closed region R in D is bounded as* $f(z)$ *varies over the functions of the family unless the constant* a *is included in the class of limit functions of the family.* These are some of the interesting results derived from the theory of normal families.†

5·6. **Schottky's theorem.** Let $f(z)$ be regular in the circle $|z|<R$. Let $f(z)$ omit to take the two values 0 and 1 in the circle. Let $|f(0)|<\delta$ and $|1-f(0)|>\delta$.

Then $$|f(z)|<M(r,\delta),\ |z|\leq r<R \qquad \dots (9)$$

where $M(r,\delta)$ *depends only on* r *and* δ *and not on the particular function satisfying the conditions.* This result is known as Schottky's theorem. It has an intimate connection with the theory of normal families. In fact, Schottky's theorem along with the results of § 5·1 can be used to prove the test of exceptional values given in § 5·2. Conversely, if the test of 5·2 be assumed, Schottky's theorem‡ could be derived by using the properties mentioned in § 5·5.

5·7. In conclusion we wish to mention that the definition and properties of normal families can be extended to a family of functions *meromorphic* in a domain D (that is, having only poles in D). The theory of normal families have important and extensive applications in the theory of integral and meromorphic functions and in the subject of conformal representation.** In this connection it may be

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* See § 2 for the definition of an infinite limit function.

† For these and other results see P. Montel, last cited. Also, G. Valiron, *Sur les valeurs exceptionnelles des fonctions méromorphes et de leurs dérivees* (Actualites Scientifiques et Industrielles, No. 570, 1937.)

‡ See P. Dienes and E. C. Titchmarsh last cited.

** See C. Caratheodory, "*Conformal Representation*" (Cambridge Tracts).

pointed out that the proof * of the test of exceptional values for normal families given in § 5·2 is essentially based on the topological fact that a *simply connected domain having at least two boundary points can be conformally represented on a finite circle.*

* See P. Montel, last cited.

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Further references will be found in (4) and (10)

SOME DIOPHANTINE PROBLEMS AND NUMERICAL IDENTITIES

BY

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1. I give firstly a solution of

$$A_1, A_2, A_3, A_4, A_5, A_6, \overset{4}{=} B_1, B_2, B_3, B_4, B_5, B_6 \qquad \ldots \quad (1{\cdot}1)$$

by which is meant that the sum of the n^{th} powers of the numbers A_r is equal to the sum of the n^{th} powers of the numbers B_r for $n = 1, 2, 3,$ and 4.

THEOREM. *Given that*

$$\left.\begin{aligned} &x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = s \\ \textit{and} \qquad &x_1 \cdot x_2 \cdot x_3 = y_1 \cdot y_2 \cdot y_3 \end{aligned}\right\} \qquad \ldots \quad (1{\cdot}2)$$

then we have also

$$x_1, x_2, x_3, s - y_1, s - y_2, s - y_3 \overset{4}{=} y_1, y_2, y_3, s - x_1\ s - x_2, s - x_3$$

A solution for the equations (1·2) is given by

$$\begin{aligned} x_1 &= (ab - cd)\,a & y_1 &= (ab - cd)\,c \\ x_2 &= (ab - cd)\,b & y_2 &= (ab - cd)\,d \\ x_3 &= (a + b - c - d)\,cd & y_3 &= (a + b - c - d)\,ab \end{aligned}$$

Example: Take $a = b = 3$, $c = 4$, $d = 1$ and we have $15 + 15 + 4 = 20 + 5 + 9$; $15 \cdot 15 \cdot 4 = 20 \cdot 5 \cdot 9$; $s = 34$ and we get

$$15, 15, 4, 14, 29, 25 \overset{4}{=} 20, 5, 9, 19, 19, 30$$

2. Under certain circumstances we get solutions of (1·1) with only 5 numbers on each side. Thus $1 + 9 + 10 = 2 + 3 + 15$ and $1 \cdot 9 \cdot 10 = 2 \cdot 3 \cdot 15$. Here $s = 20$ and the above method gives

$$1, 9, 10, 18, 17, 5 \overset{4}{=} 2, 3, 10, 11, 15, 19$$

from which 10 may be struck off on both sides.

3. Let us further impose the condition that $A_1\ A_2\ A_3$ and $B_1\ B_2\ B_3$ are to be perfect squares, and write $A_r = M_r^2$, $B_r = P_r^2$, $(r = 1, 2, 3)$,

By solving the equations †

$$M_1^2+M_2^2+M_3^2=P_1^2+P_2^2+P_3^2$$
$$M_1^2\cdot M_2^2\cdot M_3^2=P_1^2\cdot P_2^2\cdot P_3^2$$

we obtain the required solutions.

Example: $69^2+18^2+19^2=57^2+46^2+9^2=5446$

and $69^2\cdot 18^2\cdot 19^2=57^2\cdot 46^2\cdot 9^2$

whence we have

$$69^2,\ 18^2,\ 19^2,\ 2197,\ 3330,\ 5365 \overset{4}{=} 57^2,\ 46^2,\ 9^2,\ 685,\ 5122,\ 5085$$

4. To obtain solutions of (1·1) where A_1, A_2, A_3, B_1, B_2, B_3 are perfect cubes, we write

$$A_1=U_1^3,\ A_2=U_2^3,\ A_3=U_3^3;\ B_1=V_1^3,\ B_2=V_2^3,\ B_3=V_3^3$$

where

$$U_1=x\ (1+2x^3),\ U_2=-x^2\ (2+x^3),\ U_3=1-x^3$$
$$V_1=1+2x^3,\ V_2=x\ (2+x^3),\ V_3=x^2\ (1-x^3)$$

and x is arbitrary.

Example: Put $x=2$. We thus obtain

$$(-34)^3,\ 40^3,\ 7^3,\ 29952,\ 17039,\ 3087 \overset{4}{=} (-17)^3,\ 20^3,\ 28^3,\ -14305,\ 89039,\ 24696$$

5. To obtain solutions of (1·1) where A_1, A_2, A_3, B_1, B_2, B_3 are perfect 4th powers, we write

$$A_1=W_1^4,\ A_2=W_2^4,\ A_3=W_3^4;\ B_1=Z_1^4,\ B_2=Z_2^4,\ B_3=Z_3^4$$

We have now to obtain solutions of

$$W_1^4+W_2^4+W_3^4=Z_1^4+Z_2^4+Z_3^4$$
$$W_1\cdot W_2\cdot W_3=Z_1\cdot Z_2\cdot Z_3 \qquad \dots \quad (5\cdot 1)$$

After we have obtained a solution of the equation $C^4+D^4=E^4+F^4$ with the help of Euler's formula, we write

$$W_1=F^2,\ W_2=CE,\ W_3=DE;\ Z_1=E^2,\ Z_2=CF,\ Z_3=DF$$

which will be found to satisfy (5·1).

For example with $C=134$, $D=133$, $E=158$, $F=59$ we have $s=W_1^4+W_2^4+W_3^4=3481^4+21172^4+21014^4$. The solution of (1·1) is now $A_1=3481^4$, $A_2=21172^4$, $A_3=21014^4$, $A_4=(s-24964)^4$

$A_5=(s-7906^4)$, $A_6=(s-7847^4)$; $B_1=24694^4$

$B_2=7906^4$, $B_3=7847^4$, $B_4=(21172^4+21014^4)$

$B_5=3481^4+21014^4$, $B_6=3481^4+21172^4$.

† Vide: Moessner and Gloden: Diophantische Probleme *The Proc. of the Indian Academy of Sciences VIII, p. 203.*

Since the system of equations $G_1^n+G_2^n+G_3^n=H_1^n+H_2^n+H_3^n$; $G_1^n\,G_2^n\,G_3^n=H_1^n\,H_2^n\,H_3^n$ for $n>4$ has not been solved yet, we cannot get general solutions of (1·1) where $A_1\,A_2\,A_3\,B_1\,B_2\,B_3$ are perfect fifth or higher powers.

6. Here is an elementary solution in integers of the equation

$$x+y=u^2;\quad x^3+y^3=t^2$$

We write

$$x=(a^2+3b^2-2ab)^2-(a^2-b^2)^2$$
$$y=(a^2-b^2)(a^2-4ab+7b^2)$$
$$u=a^2-4ab+b^2$$
$$t=(a^4-2a^3b+6a^2b^2-14ab^3+13b^4)(a^2-4ab+b^2).$$

Example: Put $a=3$, $b=2$ and we have

$$56+65=11^2$$
$$56^3+65^3=671^2.$$

7. Here are some new solutions of Tarry's problem:

1, 4, 6, 8, 15, 19, 20, 20, 22, 27, 29, 34, 38, 41, 42, 57, 58, 61, 65, 70, 72, 77, 79, 79, 80, 84, 91, 93, 95, 98 $\stackrel{13}{=}$ 2, 2, 7, 9, 14, 16, 21, 21, 25, 28, 28, 30, 40, 40, 44, 55, 59, 59, 69, 71, 71, 74, 78, 78, 83, 85, 90, 92, 97, 97.

By using theorems on simultaneous identities already proved, we obtain from the above the relation

$$(2, 6, 6, 11, 15, 16, 18, 18, 21, 30, 33, 33, 34, 44, 47, 49, 52)^n$$
$$=(4, 5, 8, 9, 13, 13, 17, 23, 26, 27, 28, 32, 40, 42, 46, 51, 51)^n$$ for $n=1$ 3, 5, 7, 9, 11, 13.

NOTES AND DISCUSSIONS

On the Normalizing Factors for Legendre Polynomials

In all the treatments of Legendre polynomials known to me, the derivation of the relation

$$(1)\quad \int_{-1}^{1} P_n^2(u)\,du = \frac{2}{2n+1}$$

is fairly complicated.* I propose to give two simple proofs of (1) one based on each of the standard recursion formulas†

$$(2)\quad (n+1)\,P_{n+1}(x) - (2n+1)\,xP_n(x) + n\,P_{n-1}(x) = 0,$$

$$(3)\quad nP_n(x) = xP_n'(x) - P_{n-1}'(x).$$

Proof from (2). We use, beyond (2), only the facts that $P_n(x)$ is of degree n, and orthogonal to every polynomial of degree $<n$.

Multiply (2) by $P_{n-1}(x)$ and integrate over $(-1, 1)$. We find

$$(4)\quad (2n+1)\int_{-1}^{1} xP_n(x)\,P_{n-1}(x)\,dx = n\int_{-1}^{1} P_{n-1}^2(x)\,dx.$$

In (2), replace n by $n-1$:

$$nP_n(x) - (2n-1)\,xP_{n-1}(x) + (n-1)\,P_{n-2}(x) = 0;$$

multiply by $P_n(x)$, and integrate, obtaining

$$(5)\quad n\int_{-1}^{1} P_n^2(x)\,dx = (2n-1)\int_{-1}^{1} xP_n(x)P_{n-1}(x)\,dx.$$

Comparing (4) and (5), we have

$$(6)\quad \int_{-1}^{1} P_n^2(x)\,dx = \frac{2n-1}{2n+1}\int_{-1}^{1} P_{n-1}^2(x)\,dx;$$

and since $P_0(x) \equiv 1$, $\int_{-1}^{1} P_0^2(x)\,dx = 2$, successive applications of (6) give (1).

Proof from (3). Here we use also the fact that $|P_n(\pm 1)| = 1$.

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* See, for example, O. D. Kellogg, *Foundations of potential theory*, 1929, p. 132; M. H. Stone, *Developments in Legendre polynomials*, Annals of Math. (2), vol. 27 (1925) pp. 315-329.

† Kellogg, p. 126, (7), and p. 127, (9); Stone, p. 316, (1) and (3).

Multiply (3) by $P_n(x)$ and integrate, obtaining since $P_n(x)$ is orthogonal to $P'_{n-1}(x)$,

$$n\int_{-1}^{1} P_n^2(x)\,dx = \int_{-1}^{1} xP_n(x)\,P_n'(x)\,dx$$

$$= \Big[xP_n^2(x)\Big]_{-1}^{1} - \int_{-1}^{1} P_n(x)\,[xP_n'(x)+P_n(x)]\,dx$$

$$= 2 - n\int_{-1}^{1} P_n^2(x)\,dx - \int_{-1}^{1} P_n^2(x)\,dx,$$

and hence (1).

The proof based on (2) can be extended to general orthogonal polynomials $\phi_n(x)$, which satisfy the recursion relation*

$$\phi_{n+1}(x) - (x - c_{n+1})\,\phi_n(x) + \lambda_{n+1}\,\phi_{n-1}(x) = 0 \quad (\phi_0 = 1,\ \phi_{-1} = 0;\ \lambda_n > 0).$$

We obtain, if (a, b) is the interval of orthogonality of the $\phi_n(x)$, and $\psi(x)$ is the associated weight function,

$$(7)\quad \int_a^b \phi_n^2(x)\,d\psi(x) = [\psi(b) - \psi(a)]\,\lambda_{n+1}\,\lambda_n \cdots \lambda_1.$$

Relation (7) is well known in another form.†

Ong's Hat
New Jersey, U. S. A. E. S. PONDICZERY.

A Note on some Diophantine Equations

1. Consider the Diophantine equations:

$$x^2 + y^2 + z^2 = u^2 + v^2 + w^2, \qquad \ldots (1)$$

and

$$x^4 + y^4 + z^4 = u^4 + v^4 + w^4, \qquad \ldots (2)$$

where

$$x = y + z \text{ and } u = v + w. \qquad \ldots (3)$$

Substituting $y+z$ for x and $v+w$ for u in (1) and (2), we get

$$y^2 + yz + z^2 = v^2 + vw + w^2, \qquad \ldots (4)$$

and

$$(y^2 + yz + z^2)^2 = (v^2 + vw + w^2)^2. \qquad \ldots (5)$$

We are thus left to solve the equation (4) only.

Let $y - z = m$ and $v - w = n$;

so that

$$y = \tfrac{1}{2}(x+m),\ z = \tfrac{1}{2}(x-m);\ v = \tfrac{1}{2}(u+n), \text{ and } w = \tfrac{1}{2}(u-n).$$

* See, e.g., J. Shohat, *Théorie générale des polynomes orthogonouux de Tchebichef*, Mémorial des Sci. Math., fasc. 66 (1934), p. 24.

† Shohat, p. 25.

Substituting these values in (4), we now obtain

$$3x^2 + m^2 = 3u^2 + n^2,$$

or $$3(x+u)(x-u) = (n+m)(n-m); \quad \ldots \quad (6)$$

which is an equation of the form[1]:

$$3rs = pq \cdot \quad \ldots \quad (7)$$

As a solution of (7) we have

$$r = ab, \ s = cd; \qquad p = 3ac, \ q = bd.$$

Hence we easily obtain[2]

$$x = \tfrac{1}{2}(ab+cd), \qquad u = \tfrac{1}{2}(ab-cd),$$
$$y = \tfrac{1}{4}(ab+cd+3ac-bd), \qquad v = \tfrac{1}{4}(ab-cd+3ac+bd),$$
$$z = \tfrac{1}{4}(ab+cd-3ac+bd); \qquad w = \tfrac{1}{4}(ab-cd-3ac-bd).$$

These provide rational solutions of (1), (2) and (3), in terms of four parameters: a, b, c and d. Integral solutions are easily deduced now.

Example. Taking $a = 13$, $b = 11$, $c = 7$ and $d = 5$,

we have $x = 89$, $y = 99$, $z = -10$; $u = 54$, $v = 109$, $w = -55$.

Hence $99^h + 89^h + 10^h = 109^h + 55^h + 54^h$, $h = 2, 4$.

Taking $ab = -cd$, it can be easily shown that the equations:

$$2\ x^2 = u^2 + v^2 + w^2,$$
$$2\ x^4 = u^4 + v^4 + w^4;$$

have solutions in integers.

As an example, we have $x = 7$, $u = 8$, $v = 5$, and $w = 3$.

2. We next consider the equations[3]:

$$3\ x^2 = u^2 + v^2 + u^2, \quad \ldots \quad (8)$$
$$3\ x^4 = u^4 + v^4 + w^4. \quad \ldots \quad (9)$$

Multiplying (9) by 3 and subtracting the square of (8), we obtain $0 = (u^2 - v^2)^2 + (v^2 - w^2)^2 + (w^2 - u^2)^2$.

Hence $u = v = w = x$.

Thus the equations (8) and (9) have no non-trivial solution.

References

[1] Such equations are considered in my paper entitled "A Problem in Diophantine Analysis." See the *American Journal of Mathematics*, Vol. 56, (1934), pp. 269—74.

[2] For another solution see *Jahresbericht*, 49, 1939, 9—11.

[3] Suggested by Alfred Moessner.

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ASTRONOMICAL NOTES

Planet Notes from June to August 1939

Note.—All times hereunder used are I. S. T. and the co-ordinates of planets given are for 5 hours 30 minutes I. S. T.

Sun. During the period the Sun moves from Taurus to Leo with increasing North declination and attains its maximum North Declination on 22nd June the date of Summer Solstice which marks the Summer season. After this the Sun moves with decreasing North Declination. On 5th July the earth is in aphelion.

The Sun's position on the 1st of each month will be as follows:—

		June	*July*	*August*
R. A.	...	4 h 32 m	6 h 36 m	8 h 41 m
Decl.	...	21° 54′ N	23° 11′ N	18° 19′ N

Moon. The following phases of the Moon will occur.

	Full Moon	*Last Quarter*	*New Moon*	*1st Quarter*
June	2nd at 8-41 A.M.	10th at 9-37 A.M.	17th at 7-7 P.M.	24th at 10 5 A.M.
July	1st at 9-46 P.M. 31st at 0-7 P.M.	10th at 1-19 A.M.	17th at 2-33 A.M.	23rd at 5-54 P.M.
August	30th at 3-39 A.M.	8th at 2-48 P.M.	15th at 9-23 A.M.	22nd at 2-51 A.M.

The Moon will be nearest the earth on 20th June, 18th July and 15th August and will be away on 8th June, 5th July and 29th August.

Positions of the Moon.

		1st June	*1st July*	*1st August*
R. A.	...	15 h 37 m	18 h 2 m	21 h 7 m
Decl.	...	17° 43′ S	19° 21′ S	11° 26′ S

Mercury. Early in June and towards the latter part of August the planet is poorly placed for observation as a morning object, but during July it is favourably seen as an evening object. Mercury is stationary on two occasions viz., 26th July and 20th August. It is at its greatest elongation East on July 13th and West on 28th August. The planet is in superior conjunction with the Sun on 7th June and in inferior conjunction on 10th August. It is also in conjunction with the Moon on the 18th July.

Position of the planet.

		1st June	*1st July*	*1st August*
R. A.	...	3 h 59 m	8 h 15 m	9 h 37 m
Decl.	...	20° 13′ N	21° 22′ N	9° 37′ N

Venus. The course of the planet during the period is from Taurus to Leo and its motion direct. Throughout the period Venus is a morning object and by the end of August it nears the Sun and is hardly visible. The planet is in conjunction with Uranus on 5th June, with Moon on the 15th June and with Regulus on the 25th August.

Position of the planet.

		1st June	*1st July*	*1st August*
R. A.	...	2 h 47 m	5 h 17 m	8 h 1 m
Decl.	...	14° 29′ N	22° 31′ N	21° 13′ N

Mars. Mars is in Capricornus throughout. During the early part of June the planet rises by 9 P.M. and it rises earlier day by day so that by the end of August it is seen till 3 A.M. On June 24th Mars is stationary and its motion retrograde and again it is stationary on August 24th when its motion is changed direct. It is in opposition with the Sun on July 23rd and is in conjunction with the Moon on 6th June, 3rd, 30th July and 26th August.

		1st June	*1st July*	*1st August*
R. A.	...	20 h 19 m	20 h 32 m	20 h 3 m
Decl.	...	22° 32′ S	24° 5′ S	27° 3′ S

Jupiter. This planet remains in the constellation Pisces. Early in June Jupiter rises after midnight and by the end of August it rises by 8 P.M. On July 30th the planet is stationary and then it moves with retrograde motion. This planet is in conjunction with the Moon on 13th June, 11th July and 7th August.

		1st June	*1st July*	*1st August*
R. A.	...	0 h 15 m	0 h 29 m	0 h 35 m
Decl.	...	0° 22′ N	1° 46′ N	2° 9′ N

Saturn. Saturn is in Aries throughout and its motion direct till it reaches the stationary point on 14th August and then its course is changed to retrograde. During the beginning of June it is visible as a morning object and by the end of August it rises by 8 P.M.

The planet is in conjunction with the moon on 13th June, 11th July and 7th August.

Position of the planet.

		1st June	*1st July*	*1st August*
R. A.	...	1 h 44 m	1 h 54 m	2 h 0 m
Decl.	...	8° 15′N	9° 6′N	9° 29′ N

Uranus. Uranus appears as a morning object on the Eastern sky. It is stationary on the 28th August.

Position of the planet.

		1st June	*1st July*	*1st August*
R. A.	...	3 h 7 m	3 h 13 m	3 h 17 m
Decl.	...	17° 12′ N	17° 36′ N	17° 52′ N

Neptune. This planet rises just an hour after midnight during June and is visible throughout the night by the end of August.

It is stationary on the 2nd June.

Position of the planet.

		1st June	*1st July*	*1st August*
R. A.	...	11 h 27 m	11 h 28 m	11 h 30 m
Decl.	...	4° 50′ N	4° 44′ N	4° 27′ N

H. SUBRAMANI IYER.

GLEANINGS

At least one of the great mechanical engineers of the cosmos seems to have convinced himself that the models were real things. It was that persistent confusion of the map with the thing mapped, and it was a perfect, hideous picture of the age which brought it forth.

Optimism hurdled all previous records; the riddle of the universe was about to be solved — by machinery. One more gadget here, a nut or two there, and the gigantic machine would continue to function till God or the second law of thermodynamics stopped it.

Then some careless mechanic dropped a monkey wrench. There was a stripping of gears, a last terrific threshing of broken piston rods, a mad scurrying of teetering wheels into the infinite, and the whole machine went to complete and final smash hortly safter the year 1900.

From E. T. BELL, *Numerology.*

REVIEWS

RAYMOND CLARE ARCHIBALD: *Outline of the History of Mathematics.* Fourth Edition, revised and enlarged. Price 50 cents. 1939. (The Mathematical Association of America, Inc., Oberlin, Ohio).

The need for propagating a knowledge of the History of Mathematics among students is now fairly well recognised. For teachers and research-workers, the knowledge is almost indispensable. The history, however, is not merely a collection of theorems chronologically arranged. It should give a satisfactory account of the men (or women), behind the theorems and also how they fit in with the period and the country in which they lived. For a real understanding of the spirit and scope of mathematics, its continuity, its inevitability and grandeur, the history of the subject is essential. Since the days of Ball's History of Mathematics, several historico-mathematical researches in Greek, Babylonian and Indian mathematics, as well as the immense growth of the subject itself in recent decades have added to the wealth of available material, from which only an imperfect selection can be made by any one trying to give an outline such as the one under review. In its original form the outline contained the substance of two lectures delivered by Prof. R. C. Archibald before Engineering teachers about eight years ago. It is evident that it has been a fairly popular book all these years and we congratulate the author on the present edition, which is an appreciable extension of the earlier editions with a number of changes and corrections due notably to the influence of Otto Neugebauer, the famous interpreter of Babylonian Mathematics.

The sages of ancient India condensed all important lore in the form of *Sutras* to facilitate their learning and retention. The same object is now achieved through Primers and Outlines. The present Outline is an effective summary of the History of Mathematics in two sections of about 25 pages each. The first section adopts the national basis and deals with Babylonian and Egyptian mathematics, Greek mathematics, Asiatic mathematics (the mathematics of the Hindu, Arabic, and Persian but not Chinese and Japanese who are now more interested in mutual destruction than in mathematics), and European mathematics of the period 1200–1600. By way of detail, I may mention that Hindu mathematics is finished off in 36 lines

with the mention of two names Brahmagupta and Bhaskara. Following probably T. L. Heath, the author attributes the general solution of $x^2 - Ay^2 = 1$ in integers, wrongly to Brahmagupta also. (For a proper idea of the Indian method, and its implications vide *Current Science*, Vol. VI, No. 12, June 1938. p. 602).

The second section deals with the history chronologically by centuries, stopping short however of the twentieth. Here the author has imposed upon himself certain limitations in regard to the subject-matter in order to sustain the interest of the general reader.

The book concludes with a valuable literature list and notes. This list is not, however, exhaustive, as there are many important omissions. We commend the book heartily to all students of mathematics. May I end this review with the enthusiastic utterance of the Vedic mathematician, more than 3000 years ago, in the following terms:

'Yathā sikhā mayurāṇām nāgānām maṇayo yathā
Tadvad vedānga Sastraṇām gaṇitam mūrdhani sthitam.'

'Even as the crests are to peacocks, and the hood-gems to serpents, so the Science of computation stands at the head of scientific accessories to the Vedas.'

The history of a science which has claimed such enthusiastic devotees from the very beginning of its existence ought to have a great cultural value.

A. A. K.

GEORGES HOSTELET. Le Progres de L'esprit II. *Les fondements expérimentaux de l'analyse mathématique des faits statistiques.* Pp. 70. 15fr. (1937). III. *Le concours de l'analyse mathématique a l'analyse expérimentale des faits statistiques.* Pp. 69. 15fr. (1938) Actualités scientifiques et industrielles, 552, 585. (Hermann, Paris).

The first of these pamphlets is a development from a communication on the experimental foundations of the mathematical analysis of statistical facts presented to the International Institute of Statistics at the Athens session. It also gives the author's replies to the comments raised during the session. The main point emphasized is that it is indispensable for the algebraists to elucidate clearly the experimental postulates underlying the formulæ offered by them for the benefit of the statistician, such as the arithmetic

mean, the standard deviation and the coefficient of binomial correlation. The statistician should be informed of the limiting conditions for the experimental validity of the mathematical formulæ and therefore of the conditions to be realised by the statistical facts in order that the formulæ may be legitimately applied to them. The author examines the experimental foundations of probability and chance, as well as Pearson's theory of contingency, which is deeply criticised as lacking in experimental validity.

The second pamphlet attempts to complete the train of thought pursued in the first, by elaborating the implications of the three modes of scientific investigation, viz., experimental empiricism, abstract deduction, and experimental abstraction. In the second part the author embarks on the discussion of the positions of M. Reichenbach, M. de Broglie, and M. Barzin towards the notions of causality and determinism, the comparison of philosophic and scientific attitudes and the role of scientific methodology.

In the view of the author's methodology, the experimental method may be said to be centripetal and the mathematical method centrifugal. Scientific exploration is a collaboration of the activities of intuition and reasoning controlled by experimental analysis.

Intuition by itself is fallible and therefore needs control, it is confused and therefore should be made more precise. The scientific method consists essentially in a certain mode of elaboration and verification of intuitions which spring in gifted minds in the course of their explorations of physical, biological, psychological and social facts. The interpretation of facts progresses, beginning from intuition, by experimental analysis controlled by legitimate abstractions and successive re-integrations of real conditions.

The papers are well worth the attention of all those who are interested in the scientific methodology of the modern continental school, of which one of the illustrious exponents is M. L. Brunschvicg, the editor of the series 'Le Progres de l'Esprit'.

Mysore. A. A. K.

BRIJ MOHAN: *The Intermediate Solid Geometry*, Mohan & Co., Moradabad. Pp. 120, Price Re. 1-4-0.

In this book which is designed to meet the requirements of the Intermediate in Arts students of Indian Universities, the author has presented in a concise form the contents of the first 21 propositions

of Euclid's Book XI. Among solids the Prism, Pyramid and general Polyhedra are treated and also the mensuration of the Cylinder, Cone and Sphere. Examples from various University papers are interspersed throughout the book, but one feels that the total number of exercises for the learner might have been increased with advantage. The matter is carefully arranged and presented and the diagrams and printing are fairly good.

K. RANGASWAMI.

JAHNKE-EMDE: *Functionentafeln mit Formeln und Kurven · Tables of Functions with Formulae and Curves*, Third revised edition, B. G. Teubner, 1938, pp. 305.

The second edition of this book (in two languages German and English) which is both a compendium of important formulae and a table of functions, appeared in 1933 and was reviewed in detail in *The Mathematics Student* Vol. III, pages 76-77. The most valuable feature of that edition was the graphical representation of functions by means of "relief surfaces"—a method which gives a wonderful insight into the characteristic features of the function as a whole. This is naturally retained in this edition and extended to the study of other functions such as the Mathieu functions for which the x, q surfaces are given for $ce_0(x, q)$, $ce_m(x, q)$, $se_m(x, q)$ $m=1, 2$.

The new functions included in this edition are the confluent hypergeometric function, the Mathieu functions associated with the elliptic cylinder, and the Lommel-Weber and Struve Functions of orders zero and one. The elaborate treatment of Bessel functions is retained and improved by bringing the Debye series into a more convenient form. In the complete elliptic integrals of the first and second kind, formulae and tables have been added for other than Legendre's standard form. On the other hand, the material of the first 75 pages of the old edition dealing with tables of powers, complex numbers, cubic equations and the elementary transcendental functions have been omitted. It is proposed to amplify this material and issue it in separate book form as there must be many who will use this part without being interested in the higher functions.

The opportunity afforded by the new edition has been utilised to being the references to the literature on the several functions up to date.

A. N. RAO.

BOOKS RECEIVED FOR REVIEW

Actualites Scientifiques et Industrielles : Hermann & Co., Paris.

No. 649. SCHWERDTFEGER : *Les fonctions de matrices.*

No. 650. CHAIT : *Essai d' explication des crises econometriques par l' econometrie.*

No. 657. BIERNACKI : *Les fonctions multivalentes.*

No. 660. VASILESCO : *La notion de point irrégulier dans le probleme de Dirichlet.*

No. 733. MANDELBROJT : *La régularisation des fonctions.*

No. 734-740. Colloque consacre a la Théorie des probabilités et presidé par M. Maurice Frechet.

No. 734. Conférences d' introduction.

No. 735. Les fondements du calcul des probabilités.

No. 736. Les sommes et les fonctions de variables aléatoires.

No. 737. Le principle ergodique et les probabilités en chaîne.

No. 738. Les fonctions aléatoires.

No. 739. Conceptions diverses.

No. 740. La statistique mathématique.

S. R. Gupta : Elementary Analytical Dynamics of a Particle, Atma Ram and Sons, Lahore, 1939.

ANNOUNCEMENTS AND NEWS

Mr. S. Subramaniam, M.A., Statistician to the Economic Adviser to the Government of India, Delhi, has been elected a member of the Society.

Arrangements for the Hyderabad Conference of the Indian Mathematical Society are in full progress. Dr. N. R. Sen, Ghosh Professor of Applied Mathematics, Calcutta University, will deliver the Inaugural Address.

Prof. D. D. Kosambi of Fergusson College will open the Symposium on Generalised Geometry and Dr. S. S. Pillai the symposium on Waring's Problem.

Popular lectures will be delivered by (1) Prof. M. Abdur Rahman Khan of Hyderabad (Deccan) on an Astronomical subject, (2) Prof. A. Narasinga Rao of Annamalainagar on Mathematical Puzzles and Amusements, and (3) Dr. T. Vijayaraghavan of Dacca.

The Osmania University hopes to be able to provide boarding and lodging for all delegates free of charge.

Papers intended for the conference should be sent (with 2 brief abstracts) to Prof. A. Narasinga Rao, Annamalainagar, before the end of October.

Dr. B. Ramamurti, M.A., D. Sc. (Mad.), lecturer in Mathematics, Annamalai University, has been appointed Professor and Head of the Mathematics Department, Government College, Ajmere.

Dr. V. Ganapati Iyer, M.A., D.Sc. (Mad.), and Mr. V. Seetharaman, M.A., M.Sc. (Annamalai), have been appointed Lecturers in Mathematics, Annamalai University, Annamalainagar.

The Semicentennial Celebration of the American Mathematical Society

We offer our hearty felicitations and good wishes to the American Mathematical Society which celebrated its Golden Jubilee at its birthplace, Columbia University, New York City, during September 1938. The main feature of the meeting was a series of ten invited addresses of which three by Profs. R. C. Archibald, G. D. Birkhoff and E. T. Bell dealt with the history of the Society and the development of mathematics in America, and the other seven by Profs. G. C. Evans, E. J. McShane, J. F. Ritt, J. L. Synge, T. Y. Thomas, Norbert Weiner and R. L. Wilder with specific mathematical topics. These, after amplification, have been published in two volumes under the general heading of Semicentennial Addresses. There were also valuable collections of manuscripts, rare editions, instruments, models, charts etc. on exhibit. The Founder of the Society Prof. Thomas Scott Fiske was presented, after the Gala dinner held in the Hotel Astor, with a beautifully illuminated testimonial containing a greeting of appreciation and affection prepared by order of the council and signed by the President and the Secretary.

The Nobel Prize in Physics has been awarded to Prof. Enrico Fermi of the University of Rome.

Prof. James Pierpont, since 1933 Professor Emeritus of Mathematics at Yale University and well known author of books on the Theory of Functions, died in December 1938.

SOLUTIONS TO QUESTIONS

Question 1768

(R. RAMANUJAM):—Construct geometrically a triangle given the lengths of the internal bisectors of the triangle.

Remarks by Johannes Mahrenholz

The problem depends on an equation of degree 10 and is not soluble by means of ruler and compasses. See the following references:—

(1) Terquem, Considérations sur le triangle rectilgne, *Nouvelles Annales de Mathematiques*, 1 (1842) p. 87.

(2) V. Reuthe Fink, *Crelle's Journal*, 26, p. 273.

(3) Bützberger, Ein mit der Theorie algebraischer Flächen zusammenhàngendes planimetrisches Problem. *Dissertation*, Bern, 1889.

(4) F. T. Van der Berg, *Nieuv Archief voor Wiskunde*, 16 (1889), p. 179.

(5) P. Barbarin, *Mathesis*, 16 (1896), p. 143.

(6) Korselt, Über die unmöglichkeit der Konstruktion eines Dreiecks aus der drei Winkelhalbierenden, *Zeitschrift für Mathematischen und naturwissenschaflischen Unterricht*, 28 (1897), p. 81.

(7) Heymann, Zum problem der Winkelhalbierenden, *Ibid* 28 (1897), p. 165.

(8) Korselt, Über das Problem der Winkelhalbierenden, *Zeitschrift für Mathematik und Physik*, 42 (1897).

(9) Schmidt, Das problem der Winkelhalbierenden, *Jahresberiicht Kotummal* Oberrealschule, Eger, 1902, 1903, 1904.

(10) Schmidt J. Über eine für die Dreieiksgeometrie wichtige Gleichung seehsten Grades, *Archiv für Mathematik und Physik*, 3 Recte, 13 (1908), p. 108.

The Question is also found as problems in:—

(11) *Rivista Matemetica* of Peano, 2, p. 34 (Rosotti).

(12) *Nouvelle correspondence de Mathematique*, 1 (1875), p. 208 and 3 (1877), p. 32 (Brocard).

(13) *Zeitschrift für mathematischen und naturwissenschaflischen Unterricht*, 41 (1910), p. 218 (Degel).

THE MATHEMATICS STUDENT

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PROBLEM OF TIME IN PHYSICS AND MATHEMATICS

BY

SUKUMAR RANJAN DAS, M.A., Ph.D.

Time has been regarded as an all-powerful agent conditioning the exterior world by the scientists whenever they had set upon pondering over its nature and functions. After a long period of philosophical speculations the real age of scientific thinking in Europe began with Leonardo da Vinci (1452-1519), who was a universal genius. He held that whatever could be measured could be known, what could not be measured could not be known and that the relations of motion could be reduced to mathematical formulae; hence all occurrences could be explained in terms of motion and its laws. These laws which form the basis of the study of mechanics and the measurement of motion, time and space were discovered and formulated by Leonardo da Vinci. More effective was the work of Nicholaus Copernicus (1473-1543) who reached his heliocentric theory rather by speculation than by observation, but it induced thoughtful people to abandon the geocentric theory of the universe. Just after a short time Tycho Brahe (1546-1601) made the most important advance in the accuracy and number of astronomical observations leading to the measurement of time. By way of compromise between the heliocentric and geocentric theories he suggested the possibility that the sun and the moon move round the earth, while the other planets move round the sun.

About this time Galileo Galilei (1564-1641) proved the correctness of the Copernican Theory by his telescopical observations and created the scientific theory of motion by his laws of projectiles, falling bodies and the pendulum. Galileo gave to motion and to time a more precise character than had been allotted by previous thinkers. He undertook to describe more exactly how motion took place, and to give an exact measurable and mathematical character to time. According to Galileo all motions involved certain units of space or

distance covered in certain units of time. This brought space and time into prominence and gave them recognition from a new point of view. Time was no longer the mere measure of motion, but rather something independent of motion, but measured by motion.

The correctness of the Copernican Theory was further proved by John Kepler's laws of planetary movement. Kepler (1571-1630) not only dealt with the geometrical and temporal aspects of planetary motion, but for the first time in the history of human thought attempted to give an account of the mechanics of the movements of the planets.

After a short time René Descartes (1596-1650) came in the field. He looked upon Time as arising from motion, and upon our temporal experience as due to a comparison of the durations of certain regular motions, and as a manner of conceiving of duration in general. Motion was defined by him as "the transporting of one part of matter or of one body from the vicinity of those bodies that are in immediate contact with it, or which we regard at rest, to the vicinity of other bodies." Descartes raised the problem of the relativity of movement. According to Descartes rest and immobility are, and in the nature of things can only be, relative; there are no fixed points in the universe. Descartes' arguments for motion and rest as mutually dependent on systems of reference, and the grasp of the fact that no void or stable point exists by which absolute movement may be calculated, were valuable anticipations of some fundamental points involved in the Theory of Relativity. But in spite of Descartes' interest in motion and its problems, he appears to have been singularly negligent of time as a factor. He drew a distinction between time and duration and suggested that time was only a mode of thinking duration.

However, the latter half of the seventeenth century was marked by great advance in physics and astronomy. At Cambridge, Barrow, who was the predecessor and teacher of Newton, came to treat the problem of time in a new manner. He did not consider time to be identical with motion, nor did he think that time implied motion. He considered that the flow of time was independent of both motion and rest. In order to determine time, we must choose, said Barrow, some mobile which, so far as concerns the periods of its motion, keeps constantly an equal impulse and covers an equal distance. He claimed that time was not motion, but was only measurable by motion. He elaborated at a great length the discussion of the relation of time and motion. But, after all, Barrow formulated a

definite view of time as a mathematical quantity and was the first to discuss time fully and to carry that discussion, a stage further than the reiteration that time was the measure of motion. He prepared the way not only for his pupil, Sir Issac Newton, but for that very criticism of Newton which came later in the new physics leading up to the doctrine of Relativity.

Thus Newton (1642-1727) came to the field, well equipped with his great teacher's knowledge of the problem, and gave a new tone to orthodox physics which dominated scientific thought for the next two centuries. Newton conceived Real Time as an Absolute, an independent, homogeneous medium, and distinguished this Time, i.e., Absolute Time, which he called duration, from the relative "popular" time, which is a merely sensible and external measure, like hour or month. He defined absolute, true mathematical Time as something which in itself and from its own nature flows equally without relation to anything external. Such absolute Time is a constant, according to Newton. It must be admitted that such a time can be conceived, but whether it is real is quite another matter. It may be a fictitious conception, and this the modern Theory of Relativity asserts. In any case, Newton's Absolute Time presents initial difficulties, if it be true; and this Newton himself admitted when he says that "possibly there is no equable movement that can serve as an exact measure of time". The Absolute Time, therefore, cannot be known or measured, and so far as human perception is concerned must remain unknowable. With regard to the consideration of time-measurement Newton took the earth rotating in absolute space as his fundamental clock. Theoretically even this was not precisely accurate for measuring his "absolute time" unless the whole gravitational force on the earth passed accurately through the centre of mass; but even so, any discrepancy arising from this would not be serious, and could be calculated and allowed for by astronomers after observation over a sufficient number of years. Hence Newton's conception of Time, though severely criticised by the adherents of the Theory of Relativity, has not been discarded by mathematicians as it is sufficient for the purpose of mathematical calculations.

Even among Newton's contemporaries there were many who criticised Newton's Absolute Time and argued that all motion was relative and not absolute. They were of opinion that this true, absolute and mathematical Time was an unnecessary conception, unessential for everyday life and unsuited to the physicist. The Newtonian conception of Absolute Time was severely criticised by

Leibnitz (1646-1716) who though recognised its 'ideal' character in the sense that it was as independent of objects as number was independent of things numbered, yet regarded Absolute Time with its flowing moments as a fiction, and maintained a Relational Theory. Time was for him purely relative and was an order of succession. The criticisms and discussions about Newton's Absolute Time went on for more than a century and the problems of time-measurement, sometimes grasped for solution, gradually developed into a scientific discussion which culminated, later on, in the formulation of the Theory of Relativity.

When the task of time-measurement is undertaken, we have always to resort to something other than time, usually to some movement or set of movements in space. Men employ and have employed in this connection various and varying devices, such as the burning of a candle, the operation of a sand-glass, the shadow of a sun-dial, water-clocks, mechanical clocks, and chronometers, which all measure time by a motion in space. The sun is a general indicator of day and night and of seasonal times. For a field wider than the solar system the velocity of light is employed. No period of time can be measured apart from such methods as these, and our correct estimate of time must, therefore, depend on our correct measurements of space, and these we know only approximately. Astronomers reckon time by the sidereal "clock" based on the time-interval between the passage and repassage of the same star across the meridian. Thus the sidereal day, or the duration of the rotation of the earth as measured by reference to the fixed stars, is the accepted unit for the measurement of time. Even this is not satisfactory, for we have no guarantee that two complete rotations of the earth about its axis have the same duration.

The difficulties inherent in the time-measurement are particularly seen when the use of the word "simultaneity" is investigated. Two sensations are simultaneous when they are perceived instantaneously and not in succession; events are simultaneous as facts of experience just in so far as they are experienced together. But the problem becomes difficult when a physical phenomenon happening at a distance is stated to be simultaneous with an event in a mind remote from the scene of the occurrence. The issue comes to a climax in the treatment Einstein gives to the problem of the determination of simultaneity. Einstein calls two events simultaneous for a given observer when they are perceived or seen at the same time by that observer while he is equidistant from both. This, of course, involves,

and depends upon, the constancy of the velocity of light, that is to say, its speed must be the same in every direction at all times. This is assumed in the definition of simultaneity, but it is an assumption based on electro-magnetic experiments, and the work of Michelson and Morley particularly. But some do not agree with Einstein that such simultaneity is relative and not absolute. The complexities of the problem arise when we endeavour to ascertain whether events which are simultaneous in a given system of reference are also simultaneous in a frame of reference in motion with respect to the first. Arguing from the definition of simultaneity which Einstein has given, he considers whether two flashes of light at points A and B on a railway line are simultaneous. People sitting in a train travelling along the rails with a constant velocity will use the train as a rigid reference-body and they regard all events in reference to the train. If an observer situated half way between A and B sees the two flashes at A and B together, then by definition they are simultaneous events. This is merely an application of the definition. The main difficulty lies in determining whether events which are simultaneous in a given system of reference are also simultaneous in another frame of reference in a state of motion relative to the given system. By examining the above problem Einstein concludes that events which are simultaneous with reference to the railway embankment are not simultaneous with respect to the train, and *vice versa* (relativity of simultaneity) and that every reference-body (co-ordinate system) has its own particular time, unless we are told the reference-body to which the statement of time refers, there is no meaning in a statement of the time of an event. The railway embankment demonstration is a popular exposition of the Special or Restricted Theory of Relativity propounded by Einstein. The whole position there taken up by him follows from, and is a natural development of, the work of Lorentz and his equations of transformation, together with the principle of constancy of the velocity of light.

The motion of time as a fourth dimension was suggested by a friend to D'Alembert as early as 1754 and brought forward by Lange and Lagrange, also by Fechner. As early as 1901, Palagui, a Hungarian, stated his "New Theory of Space and Time," and this was the groundwork of Minkowski's conception of Time as a fourth dimension and that of the Theory of Relativity. Without Minkowski's idea of time as a fourth dimension the general theory of Relativity, as admitted by Einstein, would perhaps have not seen the light at all. It was Minkowski who showed that the Lorentz-Fitzgerald

transformation is equivalent to a twisting of a set of four mutually rectangular axes in a four dimensional space. But this amalgam arrived at by Minkowski is impossible to conceive of except as a system of mathematics. We cannot realise such a world picture, for in it Time has become mathematically symbolized. The space-time of Minkowski and Einstein is but the inevitable outcome of that view of the world begun by Galileo. Time for the mathematical physicist is and must be a measurable time.

In contrast to the Special or Restricted Theory of Relativity the following statement for the general principle of relativity was made by Einstein: All bodies of reference are equivalent for the description of natural phenomena (formulation of the general laws of nature), whatever may be their state of motion. From this principle Einstein proves that in every gravitational field, a clock will go more quickly, according to the position in which the clock is situated (at rest). For this reason, it is not possible to obtain a reasonable definition of time with the aid of clocks which are arranged at rest with respect to the body of reference.

By his theory of relativity Einstein has provoked a revolution of thought in physical science. The achievements consist essentially in this; Einstein has succeeded in separating far more completely than hitherto the share of the observer and the share of external nature in the things we see happen. The perception of an object by an observer depends on his own situation and circumstances; for example, distance will make it appear smaller and dimmer. We make allowance for this almost unconsciously in interpreting what we see. But it now appears that the allowance made for the motion of the observer has hitherto been too crude—a fact overlooked because in practice all observers share nearly the same motion, that of the earth. Physical space and time are found to be closely bound up with motion of the observer; and only an amorphous combination of the two is left inherent in the external world. When the space and time are relegated to their proper source—the observer—the world of nature which remains appears strangely unfamiliar; but it is in reality simplified and the underlying unity of the principal phenomena is now clearly revealed. Thus Time and Space are entirely relative to the standards of measurement used by observers, and these standards will vary as the relative movements of the systems in which the measurements are made vary. Simultaneity, succession and such similar relations of time and space have, for the new view in contemporary science, no absolute meaning which is eternally and immutably the same everywhere in the universe; on the contrary,

these change from system to system of reference. The work of Mach, Maxwell, Faraday and Poincare, together with the work of Michelson and Morley, which was followed up by Fitzgerald and Lorentz and finally by Einstein has led to the rejection by the new physics of the Absolute Motion, Time and Space of Newton. But nevertheless for ordinary mathematical calculations Newton's Absolute Time has not been rejected and even with all the discussions on the Mathematical Theory of Relativity by Robb and Eddington, two great exponents of Einstein's Theory, physics has been brought to the border-land of metaphysics by the speculations of Whitehead, Broad, Alexander, Eddington and Russel on the problem of Time.

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SOME PROPERTIES OF DEMLO NUMBERS AND THEIR APPLICATIONS TO RECURRING DECIMALS

BY

D. R. KAPREKAR, *Khare's Wada, Devlali.*

1. The object of this paper is two-fold. Firstly I propose to use the theory of Demlo numbers to give a solution of Question No. 1758 in Vol. VI, pp. 85-86 of *The Mathematics Student.* Next, I discuss a method by which the recurring periods in the decimal representation of the reciprocals of prime numbers can be obtained by multiplication instead of the ordinary process of division. Solutions by others for the Question No. 1758 are found in the above reference.

2. As pointed out elsewhere*, a Demlo number consists of three parts, the sum of the first and the last parts being a number with a single repeated digit while the middle part is either empty or is also a number with the same repeated digit. For instance, 231768 and 23199768 are both Demlo numbers. I shall first prove two general results which are used later to discuss the problems mentioned in § 1.

2·1. THEOREM 1. *If N is a number with k digits and P a number consisting of l ($l \geqslant k$) repeated nines, then $N \times P$ is a Demlo number in which the first part is $N-1$ and the middle part consists of $(l-k)$ repeated nines.*

* *The Mathematics Student*, 6 (1938), pp. 68-70.

Proof: Let

$$N = a_0\ 10^{k-1} + a_1\ 10^{k-2} + \cdots + a_{k-1}$$

$$P = 10^l - 1 = 999 \cdots (l \text{ times}).$$

Then

$$N \times P = a_0\ 10^{k+l-1} + \cdots + a_{k-1}\ 10^l - a_0\ 10^{k-1} - \cdots - a_{k-1}$$
$$= a_0\ 10^{k+l-1} + \cdots + (a_{k-1} - 1)\ 10^l + 9 \cdot 10^{l-1} + \cdots + 9 \cdot 10^k$$
$$+ (10 - a_0 - 1)\ 10^{k-1} + \cdots + (10 - a_{k-2} - 1)\ 10 + (10 - a_{k-1}).$$

The last relation proves the theorem.

2·2. THEOREM 2. *Let N be a number having a zero in its last digit. Then in the product*

$$Z = (N-1)\ (1 + N + N^2 + \cdot + N^r),$$

the last $(r+1)$ *digits will be nines.*

Proof: We have

$$Z = (N-1) \cdot \frac{N^{r+1} - 1}{N-1} = N^{r+1} - 1$$

and the result follows at once.

2·3. As illustrations of theorem 1 we have

$$5432 \times 99999 = 543194568$$
$$673 \times 999 = 672327.$$

As illustrations of theorem 2 we have

$$139\ (1 + 140 + 140^2 + 140^3 + 140^4)$$
$$= 50722499999$$

3. In Question 1758 mentioned in § 1 we have to prove that the last nine digits in

$$(546194538)\{1 + 24580 + (24580)^2 + \cdots + (24580)^{10}\}$$

are 999977778. To prove this, we note by theorem 1 and inspection, that

$$546194538 = 5462 \times 99999$$
$$= 2 \times 2731 \times 11111 \times 9$$
$$= 24579 \times 22222$$

Hence taking $N = 24580$ in theorem 2, we conclude that the last 11 digits in

$$24579\ \{1 + 24580 + (24580)^2 + \cdots + (24580)^{10}\}$$

are nines. Hence the product required is 22222 $\{P+Q\}$ where P contains 11 zeros at the end while Q is the number $10^{11}-1=99\cdots$ (11 times). By theorem 1, the product $22222\times Q$ is equal to

$$2222199999977778$$

and since 22222 P contains 11 zeros at the end it follows that the last nine digits in the product are those stated in the problem.

3·1. We can use a similar method for the following problems as well:

(1) Prove that the last 8 digits in

$$238053\ \{1+220+(220)^2-\cdots+(220)^5\} \quad \text{are} \quad 99998913.$$

(2) Prove that the last ten digits in

$$1375961\ \{1+140+(140)^2+\cdots+(140)^{10}\} \text{ are } 9999990101.$$

4. We next proceed to discuss the second problem mentioned in § 1. The clue to the method followed in this connection is found in theorem 2. We shall first discuss a special case and then point out the general method. We shall first explain how to obtain the recurring part of the fraction $\frac{1}{7}$ by multiplication instead of division. We find the least number a such that $7a+1$ is divisible by 10. Here $a=7$. Taking $N=7a+1=50$ in Theorem 2 we find that the last $(r+1)$ digits in

$$50^{r+1}-1=49\ \{1+50+50^2+\cdots+50^r\}$$

must be nines. Since 10^6-1 is divisible by 7 we see that the recurring portion in the fraction $\frac{1}{7}$ will be the last 6 digits in

$$\frac{50^{r+1}-1}{7}=7\ \{1+50+50^2+\cdots+50^r\}$$

for $r \geqslant 6$. This is so because if we write the above equation as

$$\frac{50^{r+1}-1}{7}=a_0\,a_1\,a_2\cdots a_p$$

we get by dividing both sides by 50^{r+1}

$$\frac{1}{7}=\frac{a_0\,a_1\cdots a_p}{50^{r+1}}+\frac{1}{7\cdot 50^{r+1}}$$

and so the last 6 digits in $a_0\,a_1\cdots a_p$ must represent the recurring

part in the fraction 1/7. Taking $r=6$ and working out the multiplication $7\,(1+50+50^2+\cdot+50^6)$ in detail we get

$$\begin{array}{r}
7\\
3\,5\,0\\
1\,7\,5\,0\,0\\
8\,7\,5\,0\,0\,0\\
4\,3\,7\,5\,0\,0\,0\,0\\
2\,1\,8\,7\,5\,0\,0\,0\,0\,0\\
1\,0\,9\,2\,7\,5\,0\,0\,0\,0\,0\,0\\
\hline
\cdots\cdot 1\,4\,2\,8\,5\,7
\end{array}$$

Hence the recurring portion is 142587. Of course if we do not give any definite value for r but continue the multiplication by powers of 50 indefinitely we shall find that the portion 142587 is repeated again and again.

4·1. We now proceed to indicate the method in the general case. Let p be any odd prime number other than 5. Let a be the least integer such that $N=ap+1$ is divisible by 10. Then the number

$$\begin{aligned}N^{r+1}-1 &= (N-1)(1+N+N^2+\cdot\cdot\cdot\cdot+N^r)\\ &= ap\,(1+N+N^2+\cdot\cdot\cdot\cdot\cdot\cdot\cdot+N^r)\end{aligned}$$

is divisible by p and has the last $r+1$ digits nines in virtue of theorem 2. Since $10^{p-1}-1$ is divisible by p it follows that a number with $(p-1)$ nines will be divisible by p. Hence, as before, if $r \geqslant p-1$, the last $p-1$ digits in

$$\frac{N^{r+1}-1}{p}=a(1+N+N^2+\cdot\cdot\cdot+N^r)$$

will represent a recurring period of the fraction $1/p$. These $p-1$ digits will either be the *actual* recurring period of $1/p$ or will contain the recurring portion a *whole number of times*. In practice, it is better to allow r to be indefinitely large and then, as the multiplication is carried on, the recurring portion will appear again and again.

4·2. We can also give the value of a in the various possible cases. We can take $a=9$, 3, 7, or 1 according as the prime ends in 1, 3, 7 or 9. We shall give another illustration with $p=19$. Here $a=1$ and we have to find the last 18 digits in

$$1\cdot(1+20+20^2+20^3+\dots) \qquad \dots \quad (1)$$

Noting that at each stage in the multiplication by 20 we have to add a zero and multiply by 2, the process of writing down each product

of the form $1 \cdot 20^k$ and adding up can be carried out shortly as follows :—

$$\begin{array}{r} \overline{\cdots 2105263157894736842 1} \\ \overline{\cdots 0101001111010110000} \end{array}$$

The recurring period is 052631578947368421. In the above process, we multiply 1 the first term in (1) by 2, put the unit figure and carry the rest (here, zero). Next we multiply 2 by 2 and proceed as in ordinary multiplication. Then $2 \times 1 = 2$, $2 \times 2 = 4$, $2 \times 4 = 8$, $2 \times 8 = 16$ (here we put 6 and carry 1), $2 \times 6 = 12 + 1 = 13$ (here we put 3 and carry 1) and so on. The carrying figure at each stage of multiplication is indicated in the lower row while the figures appearing at each stage of multiplication is shown in the top row. This process can also be applied to the product for $\frac{1}{7}$, namely

$$7(1 + 50 + 50^2 + 50^3 + \dots\dots)$$

Here we first put 7 and multiply by 5 successively. Thus

$$\begin{array}{r} \overline{\cdots\cdot 7142857} \\ \overline{\cdots\cdot 214230} \end{array}$$

This method can be applied to find successive digits of any product of the form

$$a(1 + N + N^2 + \dots\dots)$$

where N is a number ending with zero. The number a may be called the first figure and the number N from which the last zero is cancelled may be called the constant multiplier and the process itself may be called the carrying forward process. The method can be applied to any odd prime other than 5.

THE CONIC IN HOMOGENEOUS CO-ORDINATES

BY

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1. In this paper I discuss the geometry of the general equation of the second degree in a system of homogeneous co-ordinates in which the circular points are given by an assigned equation of the first degree and another of the second degree.

2. Let the equation of the line at infinity be

$$ax + by + cz = 0$$

(a, b, c not being the sides of the triangle of reference), and let

$$\theta(x,y,z) \equiv \sum u_0 x^2 + 2\sum f_0 yz = 0$$

be the equation of a fixed circle in the plane.

Let the equation of a given conic be

$$\varphi(x, y, z) \equiv \sum u x^2 + 2 \sum f yz = 0.$$

Let $X = \frac{1}{2}\dfrac{\partial\varphi}{\partial x}$, $Y = \frac{1}{2}\dfrac{\partial\varphi}{\partial y}$, $Z = \frac{1}{2}\dfrac{\partial\varphi}{\partial z}$,

and $\xi = cY - bZ$, $\quad \eta = aZ - cX$, $\quad \zeta = bX - aY$,

so that

$$a\xi + b\eta + c\zeta \equiv 0 \qquad \ldots \quad (2\cdot1)$$

Also put

$$p = vc^2 + wb^2 - 2bcf,\ q = wa^2 + uc^2 - 2acg,\ r = ub^2 + va^2 - 2abh, \quad \ldots \quad (2\cdot2)$$

and

$$-a^2p + b^2q + c^2r = 2bc\lambda, \quad a^2p - b^2q + c^2r = 2ca\mu$$
$$a^2p + b^2q - c^2r = 2ab\nu \qquad \ldots \quad (2\cdot3)$$

whence we easily get

$$ap = c\mu + b\nu, \qquad bq = a\nu + c\lambda, \qquad cr = b\lambda + a\mu, \qquad \ldots \quad (2\cdot4)$$

which give

$$\begin{vmatrix} -p & \nu & \mu \\ \nu & -q & \lambda \\ \mu & \lambda & -r \end{vmatrix} = 0$$

Let Δ stand for the determinant

$$\begin{vmatrix} u & h & g \\ h & v & f \\ g & f & w \end{vmatrix}$$

The following identities can be easily proved.

$$\begin{vmatrix} u & h & g & a \\ h & v & f & b \\ g & f & w & c \\ a & b & e & o \end{vmatrix} = -\frac{1}{4a^2b^2c^2}\begin{vmatrix} o & cr & bq & a \\ cr & o & ap & b \\ bq & ap & o & c \\ a & b & c & o \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} \lambda & o & o & a \\ o & \mu & o & b \\ o & o & \nu & c \\ 1 & 1 & 1 & o \end{vmatrix}$$

i.e. $\sum Ua^2 + 2 \sum Fbc \equiv \frac{-1}{4a^2b^2c^2}(\sum a^4p^2 - 2 \sum b^2c^2qr) = \frac{1}{abc}\sum a \mu \nu$ ... (2·6)

where U, V etc., are the co-factors of u, v, etc. in Δ.

Thus the conic φ will be an ellipse, parabola, or hyperbola according as

$$\sum Ua^2 + 2 \sum bcF > = \text{ or } < 0 \text{ or } \sum a^4p^2 - 2 \sum b^2c^2qr < = \text{ or } > 0$$
$$\text{or } \sum a \mu \nu > = \text{or} < 0 \qquad \dots \quad (2·7)$$

In particular, if the conic φ is a parabola $\sum a'\lambda = 0$, and the point at infinity on it is $(1\ \lambda,\ 1/\mu,\ 1\ \nu)$.

3. Lemma.

The joint equation of the points at infinity on the conic $\varphi(x,y,z) = 0$ is

$$\sum pl^2 - 2 \sum \lambda mn = 0 \qquad \dots \quad (3·1)$$

In particular, the joint equation of the circular points I and J at infinity is

$$\sum p_o\, l^2 - 2 \sum \lambda_o\, mn = 0 \qquad \dots \quad (3·2)$$

where p_o, λ_o, etc. are the functions of the coefficients of $\theta\ (x, y, z) = 0$ similar to p, λ, etc. (2·2, 2·3)

Proof: The points at infinity P (x_1, y_1, z_1), Q (x_2, y_2, z_2) on φ satisfy the equations

$$\varphi\ (x,\ y,\ z) = 0,\ ax + by = cz = 0$$

$$\therefore \qquad qx^2 + 2\nu xy + py^2 = 0$$

and two similar equations obtained by eliminating x and y. Hence

$$\frac{x_1x_2}{p} = \frac{y_1y_2}{q} = \frac{z_1z_2}{r} = \frac{y_1z_2 + y_2z_1}{-2\lambda} = \frac{z_1x_2 + z_2x_1}{-2\mu} = \frac{x_1y_2 + x_2y_1}{-2\nu} \qquad \dots \quad (3·3)$$

whence the lemma follows. The second part follows from the first.

4. Circles connected with the triangle of reference.

Circumcircle.

Writing the equation $\theta\ (x, y, z)=0$ in the form

$$\sum ax \sum \frac{u_0}{a} x = \frac{1}{abc} \sum ap_0yz,$$

we see that the line at infinity meets the circle $\theta\ (x, y, z)=0$ in points which lie on the circum-conic $\sum ap_0yz=0$. Thus the equation of the circumcircle of the triangle of reference is

$$\sum ap_0yz=0 \qquad \dots \quad (4{\cdot}1)$$

Polar circle.

The *polar conic* $\sum t_1 x^2=0$, will be a circle if it passes through the points I (x_1, y_1, z_1), J (x_2, y_2, z_2), i.e. if $\sum t_1 x_1^2=0$, $\sum t_1 x_2^2=0$, whence $t_1 : t_2 : t_3 = a\lambda_0 : b\mu_0 : c\nu_0$.

Hence the equation of the polar circle is

$$\sum a\lambda_0 x^2=0 \qquad \dots \quad (4{\cdot}2)$$

Conditions for a circle.

Writing the equation $\varphi\ (x, y, z)=0$ in the form

$$\sum ax \sum \frac{u}{a} x = \frac{1}{abc} \sum apyz,$$

we see that the conic φ will represent a circle, if the right-hand side equated to zero represents the circumcircle. Hence

$$\frac{p}{p_0} = \frac{q}{q_0} = \frac{r}{r_0} \qquad \dots \quad (4{\cdot}3)$$

and each one of these ratios is equal to

$$\frac{\lambda}{\lambda_0} = \frac{\mu}{\mu_0} = \frac{\nu}{\nu_0}$$

The result also follows from (3·1) and (3·2)

5. The joint equation of the polars w.r.t. ϕ of the points at infinity on the conic $\psi\ (x, y, z) \equiv \sum u'x^2 + 2\sum f'yz=0$ can be written in either of the following forms:—

$$\sum p'X^2 - 2\sum \lambda' YZ=0 \qquad \dots \quad (5{\cdot}1)$$

$$\psi\ (\xi, \eta, \zeta)=0 \qquad \dots \quad (5{\cdot}2)$$

$$\sum a\lambda'\xi^2 = 0 \qquad \dots \quad (5{\cdot}3)$$

$$\sum ap'\eta\zeta = 0 \qquad \dots \quad (5{\cdot}4)$$

where p', q', r', λ', μ', ν' are functions of the coefficients in $\psi\ (x, y, z)=0$ similar to p, q etc. (2·2, 2·3)

To prove the first relation, suppose R (x', y', z'), S (x'', y'', z'') are the points at infinity on ψ. The joint equation of their polar w. r. t. φ is

$$\sum x'X \sum x''X = 0,$$

which in virtue of (3·3) reduces to

$$\sum p'X^2 - 2 \sum \lambda' \ YZ = 0$$

To prove the second result, suppose that (x', y', z') is a point on either polar. The polar of (x', y', z') is

$$xX' + yY' + zZ' = 0$$

and the point at infinity on it is (ξ', η', ζ'). This lies on $\psi = 0$, therefore $\psi(\xi', \eta', \zeta') = 0$. Hence the equation of the polars is

$$\psi(\xi, \eta, \zeta) = 0.$$

The last two results are easy deductions from $\psi(\xi, \eta, \zeta) = 0$ in virtue of $\sum a\xi = 0$

It is obvious that these lines are diameters of φ, since ξ, η, ζ vanish at the centre of φ.

The equations $\sum a\lambda'\xi^2 = 0$, $\sum ap'\eta\zeta = 0$ are polars w. r. t. φ of the points at infinity on the conics $\sum a\lambda' x^2 = 0$, $\sum ap'yz = 0$ respectively. These points also lie on ψ; it follows therefore that the three conics

$$\psi = 0, \ \sum a\lambda' x^2 = 0, \ \sum ap'yz = 0$$

have a pair of common points at infinity.

6. Asymptotes.

The polars w. r. t. φ of the points at infinity on φ are its asymptotes. Hence the joint equations of the asymptotes of $\varphi(x, y, z) = 0$ can be written in either of the forms

$$\sum pX^2 - 2\sum \lambda YZ = 0 \qquad \ldots \ (6·1)$$

$$\varphi(\xi, \eta, \zeta) = 0 \qquad \ldots \ (6·2)$$

$$\sum a\lambda\xi^2 \equiv -\sum ap\eta\zeta = 0 \qquad (6·3)$$

The three conics $\varphi(x, y, z) = 0$, $\sum a\lambda x^2 = 0$, $\sum apyz = 0$ have a pair of common points at infinity.

7. The points at infinity on $\psi(x, y, z) = 0$ are conjugate w. r. t. $\varphi(x, y, z) = 0$, if

$$\sum up' - 2\sum f'\lambda' \equiv \sum u'p - 2\sum f'\lambda = 0 \qquad \ldots \ (7·1)$$

$$\text{or} \quad \sum a\lambda p' = 0 \qquad \ldots \ (7·2)$$

$$\text{or} \quad \sum a\lambda' p = 0 \qquad \ldots \ (7·3)$$

If $R(x', y', z')$, $S(x'', y'', z'')$ are the points at infinity on $\psi=0$ and P, Q on φ, then if R and S are conjugate w.r.t. φ, $(PQ, RS)=-1$, and therefore R and S are conjugate w.r.t. every conic through P and Q, and P and Q are conjugate w.r.t. every conic through R and S. In particular, R and S will be conjugate w. r. t. the conics $\varphi=0$, $\sum a\lambda x^2=0$, $\sum apyz=0$, hence

$$\sum ux'x''+\sum f(y'z''+y''z')=0,\quad \sum a\lambda x'x''=0,\quad \sum ap(y'z''+y''z')=0$$

from which follow the relations (7·1), (7·2), (7·3). The second part of (7·1) follows from the fact that P and Q are conjugate w.r. to ψ.

8. Rectangular Hyperbola.

The conic φ will be a rectangular hyperbola, if its points at infinity P, Q are harmonically separated by I and J. Thus P and Q (I and J) will be conjugate w. r. t. every conic through I and J (P and Q). Thus replacing $\psi(x, y, z)=0$ in (7) by $\theta(x, y, z)=0$, we get the required conditions

$$\sum up_o-2\sum f\lambda_o\equiv\sum u_op-2\sum f_o\lambda=0 \qquad \dots \quad (8\cdot1)$$

$$\sum a\lambda p_o=0 \qquad \dots \quad (8\cdot2)$$

$$\sum a\lambda_o p=0 \qquad \dots \quad (8\cdot3)$$

9. Conjugate diameters.

The equation of a pair of diameters of φ can be reduced to either of the forms

$$t_1\xi^2+t_2\eta^2+t_3\zeta^2=0 \qquad \dots \quad (9\cdot1)$$

$$T_1\eta\zeta+T_2\zeta\xi+T_3\xi\eta=0 \qquad \dots \quad (9\cdot2)$$

and these lines are the polars w.r.t. φ of the points at infinity on the conics $\sum t_1x^2=0$, $\sum T_1yz=0$ respectively. The lines (9·1) will be conjugate diameters of φ, if the points at infinity P, Q on φ are conjugate w. r. t. $\sum t_1x^2=0$,

$$\therefore\quad t_1p+t_2q+t_3r=0 \qquad \dots \quad (9\cdot3)$$

Similarly, the lines (9·2) will be conjugate diameters of φ if

$$T_1\lambda+T_2\mu+T_3\nu=0 \qquad \dots \quad (9\cdot4)$$

Thus the equations

$$\sum\frac{t_2-t_3}{p}\xi^2=0 \qquad \sum\frac{T_2-T_3}{\lambda}\eta\zeta=0 \qquad \dots \quad (9\cdot5)$$

represent the conjugate diameters of φ.

10. Axes of the Conic φ.

The Axes of a conic are the unique pair of perpendicular conjugate diameters. Thus the conjugate diameters $\sum \frac{t_2 - t_3}{p} \xi^2 = 0$ which are the polars w. r. t. φ of the points at infinity on the conic $\sum \frac{t_2 - t_3}{p} x^2 = 0$ will be perpendicular, if their points at infinity which coincide with the points at infinity on $\sum \frac{t_2 - t_3}{p} x^2 = 0$ are harmonically separated by I and J, i.e., if I and J are conjugate w. r. t. $\sum \frac{t_2 - t_3}{p} x^2 = 0$ and this requires

$$\sum p_0 \frac{t_2 - t_3}{p} = 0.$$

Thus eliminating $\frac{t_2 - t_3}{p}, \frac{t_3 - t_1}{q}, \frac{t_1 - t_2}{r}$ from equations $\sum \frac{t_2 - t_3}{p} \xi^2 = 0$, $\sum \frac{t_2 - t_3}{p} p_0 = 0$ and $\sum \frac{t_2 - t_3}{p} p = 0$, we get the equation of the axes in the form

$$\begin{vmatrix} \xi^2 & \eta^2 & \zeta^2 \\ p & q & r \\ p_0 & q_0 & r_0 \end{vmatrix} = 0 \qquad \dots \quad (10 \cdot 1)$$

The equation of the axes can also be written in the form

$$\begin{vmatrix} \eta\zeta & \zeta\xi & \xi\eta \\ \lambda & \mu & \nu \\ \lambda_0 & \mu_0 & \nu_0 \end{vmatrix} = 0 \qquad \dots \quad (10 \cdot 2)$$

11. Axes of a parabola.

The polar of the point at infinity (x', y', z'), w.r.t. the parabola φ will be its axis if (x', y', z') and the point at infinity $(1/\lambda, 1/\mu, 1/\nu)$ on the parabola which coincides with the point at infinity on its axis are conjugate w.r.t. a circle, say $\sum a\lambda_0 x^2 = 0$. Thus the line

$$x'X + y'Y + z'Z = 0.$$

will be the axis of the parabola φ, if

$$ax' + by' + cz' = 0$$

$$a\lambda_0 \frac{x'}{\lambda} + b\mu_0 \frac{y'}{\mu} + c\nu_0 \frac{z'}{\nu} = 0$$

Thus the equation of the axis of the parabola φ is

$$\begin{vmatrix} X & Y & Z \\ a & b & c \\ \dfrac{a\lambda_0}{\lambda} & \dfrac{b\mu_0}{\mu} & \dfrac{c\nu_0}{\nu} \end{vmatrix} = 0$$

Or
$$\sum a \frac{\lambda_0}{\lambda} \xi = 0 \qquad \dots \quad (11\cdot1)$$

12. Director Circle.

Let I and J be the points at infinity common to all circles in the plane. The pencil of conjugate rays w. r. t. φ through I and J intersect on a conic through I and J and is, therefore, a circle. It is called the director circle. It can easily be proved that the tangents to φ from any point of its director circle are perpendicular.

Let the co-ordinates of I and J be respectively (x_1, y_1, z_1), (x_2, y_2, z_2). Suppose (x, y, z) are the co-ordinates of the point of intersection of two conjugate rays from I and J. The co-ordinates of these rays are respectively

$$(yz_1 - y_1z,\ zx_1 - z_1x,\ xy_1 - x_1y),\ (yz_2 - y_2z,\ zx_2 - z_2x,\ xy_2 - x_2y)$$

These lines are conjugate w. r. t. φ if

$$\sum U(yz_1 - y_1z)(yz_2 - y_2z) + \sum F[(zx_1 - z_1x)(xy_2 - x_2y)$$
$$+ (zx_2 - z_2x)(xy_1 - x_1y)] = 0$$

or $\quad \sum U(r_0y^2 + q_0z^2 + 2\lambda_0yz) - 2\sum F(p_0yz + \nu_0zx + \mu_0xy - \lambda_0x^2) = 0$

or $\quad \sum(Vr_0 + Wq_0 + 2F\lambda_0)x^2 - 2\sum(p_0F + \nu_0G + \mu_0H - \lambda_0U)yz = 0 \qquad \dots \quad (12\cdot1)$

The equation can be written in the standard form

$$\sum ax \sum \frac{Vr_0 + Wq_0 + 2F\lambda_0}{a} x = \frac{k}{abc} \sum ap_0yz. \qquad \dots \quad (12\cdot2)$$

where $k \equiv \sum a^2U + 2\sum bc\ F$.

If the conic φ is a parabola $k = 0$, and the equation

$$\sum \frac{Vr_0 + Wq_0 + 2F\lambda_0}{a} x = 0 \qquad \dots \quad (12\cdot3)$$

will give the directrix of the parabola φ.

Second Form.

Let the lines be the polars of the points R (x', y', z'), S (x'', y'', z'') w. r. t. φ, so that R and S are conjugate w. r. t. φ. Suppose that

the polar of R passes through I (x_1, y_1, z_1) and the polar of S passes through J (x_2, y_2, z_2). If (x, y, z) be a point on the director circle, then the following equations are simultaneously satisfied.

$$x'X + y'Y + z'Z = 0 \qquad x''X + y''Y + z''Z + 0$$

$$x'X_1 + y'Y_1 + z'Z_1 = 0 \qquad x''X_2 + y''Y_2 + z''Z_2 = 0$$

$$x''X' + y''Y' + z''Z' = 0$$

From the last three relations we get

$$\begin{vmatrix} X & Y & Z \\ X_2 & Y_2 & Z_2 \\ X' & Y' & Z' \end{vmatrix} = 0 \qquad \ldots \quad (12\cdot4)$$

$$\text{or } 0 = \begin{vmatrix} \sum x_1X & \varphi - xX & Z \\ \sum x_1X_2 & \sum x_2X - xX_2 & Z_2 \\ 0 & -xX' & Z' \end{vmatrix} = \begin{vmatrix} \sum x_1X & \varphi & Z \\ \sum x_1X_2 & \sum x_2X & Z_2 \\ 0 & 0 & Z' \end{vmatrix} - x\begin{vmatrix} \sum x_1X & X & Z \\ \sum x_1X_2 & X_2 & Z_2 \\ 0 & X' & Z' \end{vmatrix}$$

or $$\sum x_1X_2 \cdot \varphi(x, y, z) - \sum x_1X \sum x_2X$$

since the last determinant vanishes, (12·4) and this equation reduces to

$$\left(\sum ap_0 - 2\sum f\lambda_0\right) \varphi(x, y, z) - \sum p_0X^2 - 2\sum \lambda\, YZ \qquad \ldots \quad (12\cdot5)$$

The same result can be obtained from the consideration that the pair of tangents from a point on the director circle are perpendicular. Let $(x'y'z')$ be a point in the plane of the conic. The equations of the pair of tangents from (x', y', z') to φ is

$$\varphi(x, y, z)\, \varphi(x', y', z') = (xX' + yY' + zZ')^2$$

i.e. $$\sum (a\varphi' - X'^2)\, x^2 + 2\sum (f\varphi' - Y'Z')yz = 0$$

If (x_1, y_1, z_1) (x_2, y_2, z_2) be the points at infinity on these tangents

$$\frac{x_1\, x_2}{p\varphi' - \xi'^2} = \frac{y_1\, y_2}{q\varphi' - \eta'^2} = \frac{z_1\, z_2}{r\varphi' - \zeta'^2}$$

The points (x_1, y_1, z_1), (x_2, y_2, z_2) are conjugate w. r. t. the polar circle $\sum a\lambda_0\, x^2 = 0$ if

$$\sum a\lambda_0\, (p\varphi' - \xi'^2) = 0.$$

Hence $(x' \; y' \; z')$ lies on the locus

$$\varphi(x, y, z) \sum ap\lambda_0 = \sum a\lambda_0\xi^2 \qquad \dots \quad (12\cdot6)$$

which can also be written in the form

$$\varphi(x, y, z) \sum ap\lambda_0 = -\sum ap_0\eta\zeta \qquad \dots \quad (12\cdot7)$$

It may be remarked that the right-hand sides equated to zero of equations (12·5), (12·6), (12·7) represent the polars w. r. t. φ of the circular points at infinity, and the coefficients of $\varphi(x, y, z)$ on the left equated to zero are two different forms of condition for a rectangular hyperbola (8·1, 8·3).

13. Foci.

If the point (x', y', z'), of the previous section be a focus, the points (x_1, y_1, z_1), (x_2, y_2, z_2) at infinity on the tangents from (x', y', z') coincide with I and J. Thus

$$\frac{p\varphi' - \xi'^2}{p_0} = \frac{q\varphi' - \eta'^2}{q_0} = \frac{r\varphi' - \eta'^2}{r_0}.$$

Thus the foci lie on the conics

$$\frac{p\varphi - \xi^2}{p_0} = \frac{q\varphi - \eta^2}{q_0} = \frac{r\varphi - \zeta^2}{r_0} \qquad \dots \quad (13\cdot1)$$

Directrices. The directrices of a conic φ are the degenerate conics of the system generated by the given conic and the polars of I and J w. r. t. φ, viz.

$$k\varphi = \sum p_0 X^2 - 2\sum \lambda_0 YZ \qquad \dots \quad (14\cdot1)$$

where k is the parameter of the system. To determine k so that (14·1) may represent a pair of lines, we transform the above equation by the identity

$$\Delta\varphi = \sum UX^2 + 2\sum FYZ.$$

The equation (14·1) becomes

$$\sum(kU - p_0\Delta)X^2 + 2\sum(kF + \lambda_0\Delta)YZ = 0$$

This will be a product of two linear factors if

$$\begin{vmatrix} kU - p_0\Delta & kH + \Delta\nu_0 & kG + \Delta\mu_0 \\ kH + \Delta\nu_0 & kV - \Delta q_0 & kF + \Delta\lambda_0 \\ kG + \Delta\mu_0 & kF + \Delta\lambda_0 & kW - \Delta r_0 \end{vmatrix} = 0$$

i.e. $$k^2 - \delta_1 k + \delta_2 = 0 \qquad \dots \quad (14\cdot2)$$

where $\delta_1 \equiv \sum p_0 U - 2\sum F\lambda_0$ and $\delta_2 \equiv (\sum a^2U_0 + 2\sum bcF_0)(\sum a^2U + 2\sum bcF)$.

A NOTE ON THE RATIO OF THE IN- AND CIRCUM-RADII OF A TETRAHEDRON

BY

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1. In Question No. 1763, in *The Mathematics Student* (December 1937), it is required to prove that the inradius r of a tetrahedron is less than half the circumradius R. While solving this problem, I found that the maximum value of the ratio r/R for a tetrahedron is $1/3$, and for a simplex with $(n+1)$ vertices in n-dimensional space this value is $1/n$. The present note contains a simple proof of these results.

2. To prove the theorem for a tetrahedron we require the following:

Lemma: If the base ABC of a tetrahedron is fixed, and the fourth vertex D moves at a constant distance h from the plane ABC, then the maximum value of the inradius is $\dfrac{hr_0}{r_0+\sqrt{(h^2+r_0^2)}}$, where r_0 is the inradius of $\triangle$ ABC, and this value is attained when the foot of the altitude through D coincides with the in-centre of $\triangle$ ABC.

Proof: Let $\Delta_A, \Delta_B, \Delta_C, \Delta_D$ denote the areas of the faces of the tetrahedron; a, b, c the sides of ABC; and α, β, γ the trilinear co-ordinates of the foot of the altitude through D.

Then $r=3V\div(\Delta_A+\Delta_B+\Delta_C+\Delta_D)$; and since V and Δ_D are constant, r will be a maximum when $\Delta_A+\Delta_B+\Delta_C$ is a minimum.

Now $2(\Delta_A+\Delta_B+\Delta_C)=a\sqrt{h^2+\alpha^2}+b\sqrt{h^2+\beta^2}+c\sqrt{h^2+\gamma^2}$.

and $2\Delta_D=a\alpha+b\beta+c\gamma$

Using the method of undetermined multipliers, we find that the expression $\Delta_A+\Delta_B+\Delta_C$ has a stationary value, when

$$\frac{\alpha}{\sqrt{h^2+\alpha^2}}=\frac{\beta}{\sqrt{h^2+\beta^2}}=\frac{\gamma}{\sqrt{h^2+\gamma^2}}.$$

i.e. when $\alpha=\beta=\gamma=r_0$ when the foot of the altitude through D coincides with the incentre of ABC

Writing r_0+x, r_0+y, r_0+z for α, β, γ respectively, and considering the terms of the second degree in x, y, z in the expansion of $\sum a\sqrt{h^2+\alpha^2}$, we can show that for this position of the foot of the altitude, the expression $\Delta_A+\Delta_B+\Delta_C$ is a minimum and therefore r is a maximum. Also in this case

$$r=\frac{3V}{\Delta_A+\Delta_B+\Delta_C+\Delta_D}=\frac{\frac{1}{2}hr_0(a+b+c)}{\frac{1}{2}(a+b+c)\sqrt{h^2+r_0^2}+\frac{1}{2}r_0(a+b+c)}$$
$$=\frac{hr_0}{\sqrt{h^2+r_0^2}+r_0}$$

In the same way, for a simplex in n-dimensional space, it can be shown that, if n out of the $(n+1)$ vertices are fixed, and the $(n+1)^{th}$ vertex moves at a constant distance h from the $(n-1)$-flat passing through the n fixed vertices, then the maximum value of the inradius is $\dfrac{hr_0}{r_0+\sqrt{h^2+r_0^2}}$ where r_0 is the inradius of the fixed base, and this value is attained when the foot of the altitude through the $(n+1)^{th}$ vertex, coincides with the incentre of the base.

3. Now take any tetrahedron ABCD, inscribed in a given sphere of radius R. Let r_0 and R_0 be the in- and circum-radii of Δ ABC, and let I and O be the in- and circum-centres (of Δ ABC).

Take any equilateral triangle $A_0B_0C_0$ having the same circum-circle as Δ ABC, and therefore having its in- and circum-centres coincident with O.

Draw a plane through D parallel to the plane ABC, and let the perpendiculars at I and O to the plane ABC, meet the parallel plane through D in the points D_1 and D_2, and let OD_3 produced meet the sphere ABCD in D_0.

Then, the inradius of the equilateral triangle $A_0B_0C_0$ is greater than r_0, and since the expression

$$\frac{hr_0}{r_0+\sqrt{h^2+r_0^2}}=1\div\left\{\frac{1}{h}+\sqrt{\left(\frac{1}{r_0^2}+\frac{1}{h^2}\right)}\right\} \text{ increases with } h \text{ and } r_0$$

we see that $\begin{pmatrix}\text{inradius}\\ \text{of DABC}\end{pmatrix}<\begin{pmatrix}\text{inradius}\\ \text{of } D_1ABC\end{pmatrix}<\begin{pmatrix}\text{inradius}\\ \text{of } D_2A_0B_0C_0\end{pmatrix}<\begin{pmatrix}\text{inradius}\\ \text{of } D_0A_0B_0C_0\end{pmatrix}$

The ratio r/R is greater for the tetrahedron $D_0A_0B_0C_0$ than for the original tetrahedron DABC, since r/R has an upper bound which is actually attained. This means that so long as any face of a tetrahedron is not an equilateral triangle, and the foot of the corresponding altitude does not coincide with its incentre, it is possible to increase the ratio r/R.

It follows that this ratio will be a maximum when the tetrahedron is regular, and then its value is $\frac{1}{3}$.

4. The corresponding result for the general simplex $A_1A_2 \cdots A_{n+1}$ can now be proved by induction.

Let $B_1B_2 \cdots B_n$ be a regular $(n-1)$-dimensional simplex, having the same circumscribed hypersphere as $A_1A_2 \cdots A_n$. Let O be the centre of this regular simplex, and let B_{n+1} be the point, in which the normal at O to the $(n-1)$-flat $A_1A_2 \cdots A_n$ meets the circumscribed hypersphere of the simplex $A_1 A_2 \cdots A_n A_{n+1}$. Assuming the theorem to be true for an $(n-1)$-dimensional simplex, it can be shown exactly as in the case of the tetrahedron, that the inradius of $B_1B_2 \cdots B_n B_{n+1}$ is greater than the inradius of $A_1 A_2 \cdots A_n A_{n+1}$; and then it follows, exactly as before, that the ratio r/R is a maximum when the simplex is regular and then its value is $1/n$.

5. The particular case of this theorem for a plane triangle $(n=2)$ can be proved very easily by using the formula $OI^2 = R^2 - 2Rr$. This suggests an interesting question. Is it possible to obtain a similar formula (in terms of R and r alone) for the square of the distance between the incentre and the circumcentre of a tetrahedron (and of the general n-dimensional simplex)?

In Salmon's *Solid Geometry* Vol. I (sixth edition) p. 235 an expression is given for the square of the distance between these two points for a tetrahedron, but it involves the edges and the altitudes of the tetrahedron, and I was unable to derive from it another expression containing R and r alone. I would like to know if such a formula can be obtained for a tetrahedron (and a simplex), for it will provide the simplest proof of the main theorem of this note.

Wilson College, *Bombay.* — P. K. KASHIKAR.

Remarks on the above paper

The late Prof. J. H Grace has obtained the inequality given below in a paper entitled "Tetrahedra in relation to Spheres and Quadrics published in *Proc. Lond. Math. Society* Vol. 17 (1918). Let r, R be the in- and circum-radii of a tetrahedron and d the distance between the in- and circum-centres.

Then

$$(R+r)(R-3r) \geqslant d^2$$

From the above it follows that, for a real tetrahedron, $r/R \leqslant 1/3$.

Annamalainagar B. R. VENKATARAMAN

GLEANINGS

When, however, with much effort I reached the thirteenth proposition of Euclid, the utter simplicity of the subject was suddenly revealed to me. A subject which only required a pure and simple use of one's reasoning powers could not be difficult. Ever since that time Geometry has been both easy and interesting for me.

M. K. GANDHI—*The Story of my experiments with Truth*

"It became still more interesting when, after the angularities of a combination of straight lines, I learnt to portray the graces of a curve. How many properties were there of which the compasses knew nothing, how many cunning laws lay contained in embryo within an equation, the mysterious nut which must be artistically cracked to extract the rich kernel, the theorem! Take this or that term, place the + sign before it and forthwith you have the ellipse, the trajectory of the planets, with its two friendly foci, transmitting pairs of vectors whose sum is constant, substitute the − sign and you have the hyperbola with the antagonistic foci, the desperate curve that dives into space with infinite tentacles, approaching nearer and nearer to straight lines, the asymptotes, but never succeeding in meeting them. Suppress that term and you have the parabola, which vainly seeks in infinity its lost second focus, you have the trajectory of the bombshell; you have the path of certain comets which come one day to visit our sun and flee to depths whence they never return. Is it not wonderful thus to formulate the orbit of the worlds? I thought so then and I think so still."

FABRE,

(From L. G. SIMONS' *Fabre and Mathematics and other Essays.*)
Scripta Mathematica Library No. 4.

THE RULE OF SUCCESSION IN STATISTICAL INFERENCE

BY

S. SUBRAMANIAN AND S. P. SUBRAMANIAN

The problem of estimating the frequency of successes when the probability of success in a single event is given has been satisfactorily solved by Bernoulli, but the converse problem of obtaining a measure of probability from an observed statistical frequency remains unsolved even at the present day. Bernoulli did not proceed beyond suggesting a possible method by means of an inversion of his famous theorem. One has to come down to the time of Laplace to find a serious attempt at solving the problem. Laplace followed two methods of which the first assumes an inversion of Bernoulli's theorem and the second involves an application of Baye s Rule. His aim was to see if, when the frequency of successes in a given number of trials was known, the probability of success at subsequent trials could be inferred with accuracy. Laplace's second method has led to what is called* the Rule of Succession which says :—*If an event has occurred m times and failed n times under given conditions and we do not know anything more than that the probability of its occurrence when the conditions are next fulfilled is* $(m+1)/(m+n+2)$. There are many who do not accept the validity of Laplace's arguments. The lines along which the critics proceed are clearly indicated by Mr. Keynes in his *Treatise on Probability*. On the other hand there are those who agree with Laplace. Karl Pearson was among those who supported Laplace. Pearson restated Laplace's theorem and proved an extended form of his result by applying Bayes Rule and making use of Gamma functions. It is not the purpose of this note to rake up the old controversy about the validity of the rule or otherwise but only to point out how the rule as extended by Pearson can be established from very simple considerations. In Part I of this paper a general theorem is proved and in Part II, Pearson's extension of Laplace's rule is derived as a particular case of the general theorem.

* Venn: *Logic of Chance*; p. 190.

PART I

1. Let us consider the following problem :—

A bag contains n balls of which any number may be white and the rest black; P balls are drawn from the bag and of these p are found to be white. What is the probability that of R balls drawn next, r will be white?

Now, n may be finite or infinite and the drawing may be done in any one of three ways. The solution of the problem will therefore fall into various classes. Let $\mathrm{P}=p+q$, $\mathrm{R}=r+s$.

(1) *n is finite.*

(*a*) *The balls are drawn one by one but not replaced.*

The drawing of p white balls and q black balls is only one of $(p+q+1)$ equally likely cases starting from $(p+q)$ white balls and no black ball and ending with no white ball and $(p+q)$ black balls. Hence the probability of such a drawing is $1/(p+q+1)$.

Similarly the probability of drawing $(p+r)$ white balls and $(q+s)$ black balls *in any order* is $1/(p+q+r+s+1)$. The total number of balls in the two draws, namely, $p+q+r+s$ can be divided into groups of $(p+q)$ and $(r+s)$ in $(p+q+r+s)!\,(p+q)!\,(r+s)!$ ways. Of these we want only those in which the $(p+r)$ white balls are divided into groups of p and r and the $(q+s)$ black ones into groups of q and s. There are only $\dfrac{(p+r)!}{p!\cdot r!}\cdot\dfrac{(q+s)!}{q!\,s!}$ such groups. Hence the probability that the white and the black balls have followed the given order is

$$\frac{(p+r)!\,(q+s)!\,(r+s)!\,(p+q)!}{p!\,q!\,r!\,s!\,(p+q+r+s)!}\cdot\frac{1}{p+q+r+s+1}=k \text{ (suppose).}$$

Thus if π were the probability that p white and q black ones will be followed by r white and s black, then since the trials are independent

$$\frac{1}{p+q+1}\cdot\pi=k\text{; that is,}$$

$$\pi=\frac{(p+r)!\,(q+s)!\,(r+s)!\,(p+q+1)!}{p!\,q!\,r!\,s!\,(p+q+r+s+1)!} \quad \cdots \text{ (A)}$$

(*b*) *The P balls are drawn together at first and not replaced; then the R balls are drawn together*

The reasoning will be identical with that in the last subdivision and the answer will be the same.

(c) *The balls are drawn one by one and replaced after every attempt.*

In this case the problem becomes complicated and the argument of subdivision (a) is inapplicable; for, supposing that the first ball drawn is white and is replaced, before the second draw we know that there is at least one white ball in the bag whereas nothing is known about the black. Therefore we cannot say that all possible draws are equally likely.

(2) n is infinite

In cases (a) and (b) the arguments proceed as in section (1). But in case (c) it will be seen that the number of white balls may be infinite or the number of black ones or both and that in all these cases the fact that we draw a white ball and replace it does not give us more information about the white than about the black for the next drawing; the reason is that, here, the difference is between finite and infinite numbers. It may also be observed that the distinction between the cases of replacement and non-replacement arises only when the total number of balls is finite because only then the probabilities of drawing a white ball in the second trial will differ in the two cases. Thus result (A) holds good here also.

PART II

Karl Pearson's statement of Laplace's rule is as follows †:—

Let the chance of a given event occurring be supposed to lie between x and $x+\delta x$ and then if on $n(=p+q)$ trials the event has been observed to occur p times and fail q times........., the chance C_r, whatever x may be, of the event occurring r times in the second series of $m(=r+s)$ trials is given by

$$\frac{m!}{r!\,s!}\cdot\frac{\int_0^1 x^{p+r}(1-x)^{q+s}dx}{\int_0^1 x^p(1-x)^q\,dx}.$$

The method of the present paper and that of Laplace and Pearson can be related if we term the drawing of a white ball a success and that of a black one, a failure. The Laplace-Pearson method assumes that all values of x from 0 to 1 are equally likely and that x remains constant from ball to ball. In effect this is equivalent to

† Karl Pearson: *Philosophical Magazine* (1907) pp. 365-78.

assuming the number of balls in the bag to be infinite. This is case (2) of Part I of this paper. Pearson's formula for C_r can be easily verified to lead to result (A) of the paper on performing the integration.

The main difference between the Laplace-Pearson method and the one followed in this paper is in the distribution of ignorance. The former method would have it that every possible composition of the bag is equally likely whereas the latter assumes that every possible assortment of successes and failures in a given number of trials is equally likely. The latter method is more general than the former in that it provides for a limitation of the field of variation of the probability of success as also a change in the value of this probability from trial to trial. But it is unfortunate, however, that this general method should be inapplicable to the case where the probability of success may take any value within a limited field but is constant from trial to trial. Herein lies the weakness of all methods based on the principle of equal distribution of ignorance.

Conclusion

We have seen that a general rule of succession could be established by applying the principle of the equal distribution of ignorance in a manner different from that of Laplace and Pearson and that the Laplace-Pearson rule comes out as a particular case of the general rule formulated in this paper.

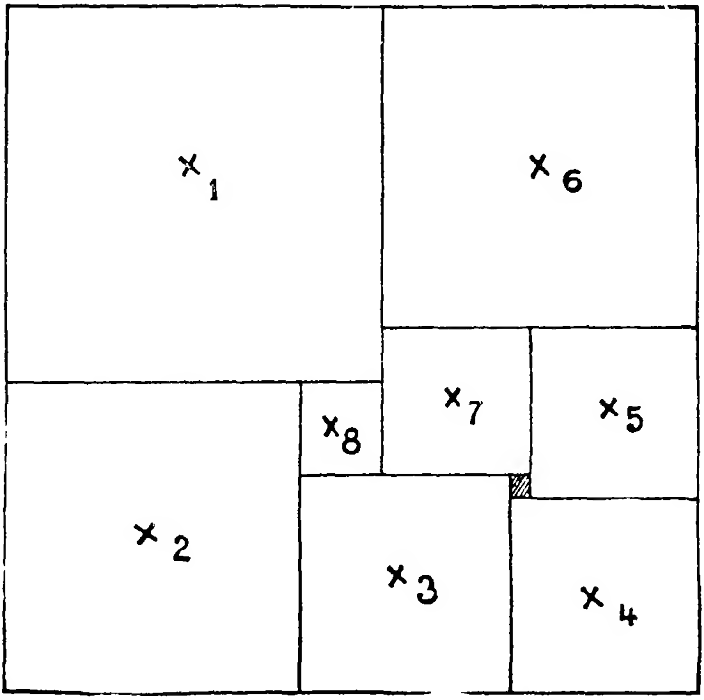
NOTES AND DISCUSSIONS

The division of a rectangle into unequal squares

§ 1. According to Steinhaus ("*Mathematical Snapshots*", published by G. E. Stechert & Co., New York) the problem of dividing a square into squares of *unequal* size, is still unsolved.

However it is possible to divide a rectangle into *nine* squares of unequal size. Steinhaus leaves the solution of this as an exercise to the reader; the end of his book contains a reference to a published solution, which is probably not available in India.

The following method seems to me a natural one for dividing a rectangle into 9 squares of unequal size. It is possible that a similar elementary method can be used to prove that it is *impossible* to divide a rectangle into *eight* unequal squares (an unsolved problem according to Steinhaus).



§ 2. Let us suppose that the lengths of the sides of the squares are x_n $(1 \leqslant n \leqslant 9)$, where no two x's are equal. Without loss of generality we may suppose that $x_9 = 1$.

The following diagram indicates a tentative division of a rectangle into squares of sides x_n $(1 \leqslant n \leqslant 9)$. In this figure all the rectangles (nine in number) are supposed to be squares of sides x_n. The shaded rectangle is a square of side x_9 $(=1)$.

If *this* division of a rectangle into unequal squares is possible then it must be possible to solve the following system of linear equations in the 8 unknowns $(x_1, x_2, x_3, \ldots, x_8)$ in *positive unequal* x's:

$$x_1 + x_2 = x_4 + x_5 + x_6$$
$$x_2 + x_3 + x_4 = x_1 + x_6$$
$$x_6 = x_5 + x_7$$
$$x_5 = x_7 + 1$$
$$x_1 = x_2 + x_8$$
$$x_3 = x_4 + 1$$
$$x_6 + x_7 = x_1 + x_8$$
$$x_3 + 1 = x_7 + x_8.$$

The reader will easily verify that this set of equations has the unique solution:

$$x_1 = 18,\ x_2 = 14,\ x_3 = 10,$$
$$x_4 = 9,\ x_5 = 8,\ x_6 = 15,$$
$$x_7 = 7,\ x_8 = 4.$$

It follows that *the whole area of a rectangle of sides* 33 and 32, *can be divided into the nine unequal squares of sides*: 1, 4, 7, 8, 9, 10, 14, 15, 18.

S. CHOWLA.

On Stirling's Approximation

I. In this note, I give an easy proof for

Theorem: $\log (n!) = F(n) + C + \theta/3(n-1)$,

where $F(n) = n \log n - n + \frac{1}{2} \log n$; $|\theta| < 1$; C is a constant.

Let $G(n) = F(n) - F(n-1) - \log n$; and $K = \sum\limits_{r=2}^{n} G(r)$.

Then $K = F(n) - F(1) - \log (n!) = F(n) + 1 - \log (n!)$. ... (1)

Further, $G(n+1) = n \log \left(1 + \frac{1}{n}\right) - 1 + \frac{1}{2} \log \left(1 + \frac{1}{n}\right)$. ... (2)

Since $\log (1+x) > x - \frac{1}{2} x^2$, from (2),

$$G(n+1) > \left(n + \frac{1}{2}\right)\left(\frac{1}{n} - \frac{1}{2n^2}\right) - 1 = -\frac{1}{4n^2}. \quad \ldots (3)$$

Again, since $\log (1+x) < x - \frac{1}{2} x^2 + \frac{1}{3} x^3 < x$, from (2),

$$G(n+1) < n\left(\frac{1}{n} - \tfrac{1}{2}/n^2 + \tfrac{1}{3}/n^3\right) - 1 + \tfrac{1}{2}/n = \tfrac{1}{3}/n^2. \quad \ldots (4)$$

From (3), (4), we get that K tends to a limit when $n \to \infty$,

$$\text{Let } A = \sum_{r=2}^{\infty} G(r).$$

Then, since $1/m^2 < 1/m(m-1) = \frac{1}{m-1} - \frac{1}{m}$,

$$|K - A| \leqslant \sum_{r=n+1}^{\infty} |G(r)| < \tfrac{1}{3} \sum 1/(r-1)^2 < \frac{1}{3(n-1)}.$$

So $K = A + \frac{\theta}{3(n-1)}$, where $|\theta| < 1$. ... (5)

From (1) and (5), we get the theorem, where $C = 1 - A$.

If we put $F(n) = n \log n - n + \frac{1}{2} \log n + \frac{1}{4n}$, similarly we can prove that $\log (n!) = n \log n - n + \frac{1}{2} \log n + \frac{1}{12n} + O(1/n^3)$.

II. By the above method, we can easily prove that

$$S(n) = \sum_{1}^{n} \frac{1}{r} = \log n + \gamma + \theta/n, \text{ where } 0 < \theta < 1.$$

Let $H(n) = \log n - \log (n-1) - 1/n$. ... (6)

Then $\sum_{2}^{n} H(r) = \log n - \sum_{2}^{n} 1/r = \log n + 1 - S(n)$. ... (7)

But, since $-\log (1-x) > x$, from (6),

$$H(n) = -\log (1 - 1/n) - 1/n > \frac{1}{n} - \frac{1}{n} = 0 \quad \text{... (8)}$$

Again $H(n) = \log \left(1 + \frac{1}{n-1}\right) - \frac{1}{n} < \frac{1}{n-1} - \frac{1}{n}$. ... (9)

From (8), (9), we get that $\sum_{2}^{\infty} H(r)$ is convergent, and let

$$D = \sum_{2}^{\infty} H(r).$$

Then $0 < D - \sum_{2}^{n} H(r) = \sum_{n+1}^{\infty} H(r) < \sum_{n+1}^{\infty} \left\{\frac{1}{r-1} - \frac{1}{r}\right\} = \frac{1}{n}$.

So $\sum_{2}^{n} H(r) = D - \theta/n$, where $0 < \theta < 1$. ... (10)

From (7) and (10), we get that

$$S(n) = \log n + 1 - D + \theta/n.$$

Annamalainagar. S. S. PILLAI.

SOLUTIONS TO QUESTIONS

Question 1702

(K. SATYANARAYANA):—The common points of the circles on the diagonals of a quadrilateral as diameters are such that each side is the ortho-line of either of the points with respect to the triangle formed by the other three. Calling these *the ortho-points of the quadrilateral*, prove that:

(1) The circumcircle of the diagonal triangle and the self-polar circles of the four triangles of the quadrilateral form a coaxal system with the ortho-points as limit points.

(2) The director circles of all inconics of the quadrilateral form a coaxal system with the ortho-points as common points.

Hence prove Gaskin's theorem.

Show also that the necessary and sufficient condition for the existence of a rectangular hyperbola in the inconic family is the coincidence of the ortho-points.

Solution by K. Rangaswami

It is well known that the director circles of conics touching the sides of a quadrilateral belong to a coaxal system Γ to which also belong the circles described on the three diagonals as diameters. Thus the director circles have in common the two ortho-points of the quadrilateral.

Now, the conics outpolar to the range is an ∞^3-system which must contain ∞^1 circles i.e., a coaxal system Γ'. Since the ortho-points P, Q lie on the director circles of all conics of the range, the tangents from P and Q to the conics are at right angles. Thus, P and Q are point circles outpolar to the range and are, therefore, the limit points of the coaxal system Γ'.

Again, the circumcircle of the diagonal line triangle of the quadrilateral and the polar circles of the four triangles formed by the lines of the quadrilateral are evidently circles outpolar to the conics of the range. Hence, these circles belong to the coaxal sytem Γ' for which P and Q are limit points and cut orthogonally the

circles of the system Γ. We have, therefore, Gaskin's theorem that any circle outpolar to a conic cuts orthogonally the director circle of the conic.

Now, among the conics of the range there will be two rectangular hyperbolas and their director circles are the point circles at their centres, say, R, R′. Thus R, R′ are the limit points of the coaxal system Γ. If, however, there is only one rectangular hyperbola in the range, R and R′ must coincide and Γ is a touching coaxal system. Hence their common points P, Q *viz.*, the ortho-points of the quadrilateral must also coincide.

Also solved by T. K. Raghavachari and M. V. Vaidyanatha Sastry.

Question 1745

(KAPREKAR D. R.):—We have $91 \times 819 = 74529$ and also $9901 \times 980199 = 9704950299$ where the corresponding numbers in the second multiplication are obtained by inserting 9 and 0 alternately before the first digit and between the successive digits of all the numbers in the first multiplication. It will be found that the multiplication is still true if we insert alternately blocks of 99 and 00 or 999 and 000 and so on. It is required to find the principle underlying the result and to find other number pairs like 91 and 819 with the same property.

Further Remarks by V. Narasimhamurthi.

A discussion of this was published by me in Vol. VI of *The Mathematics Student* pp. 83-85. The proposer remarks on p. 85 that he has not succeeded in getting any general formula for obtaining number-pairs with the property stated in the question. The number pairs, 81, 729; 71, 819; 81, 819; 92, 729; which were stated by him, satisfy the limitations of my general formula of p. 84. Hence these pairs satisfy the required property.

Mr. Kaprekar gives also two number pairs, wherein we have to regard groups of two digits as unit between which groups of 9's and 0's are to be inserted. They are

$$\left.\begin{array}{l} 940392 \times 9302 = 87\ 47\ 52\ 63\ 84 \\ 994003992 \times 993002 = 9870479\ 52063984 \end{array}\right\} \quad \text{(i)}$$

$$\left.\begin{array}{l} 970193 \times 980192 = 95\ 09\ 75\ 41\ \cdot 7056 \\ 997001993 \times 998001992 = 995009975041970056 \end{array}\right\} \quad \text{(ii)}$$

We may also insert r nines and r zeros in the gaps instead of single nines and single zeros. My solution was based on the relation

$$[x^m - a_1x^{m-1} + (a_2+1)x^{m-2} \mp \cdots\cdots] \times [x^n - b_1x^{n-1} + (b_2+1)x^{n-2} \mp \cdots\cdots]$$
$$\equiv [x^l - c_1x^{l-1} + (c_2+1)x^{l-2} \mp \cdots\cdots\cdots\cdots]$$

For this above property to hold, this identity must be satisfied for every $x = 10^t$ where $t > 1$. In that case the property in the question fails and the property in the remark can be obtained. But the upper limits of c's will be different in this case.

$$10 > a_1,\ b_1 > 0;\ 10 > a_{2r},\ b_{2r} \geqslant 0;$$
$$10 \geqslant a_{2r+1},\ b_{2r+1} > 0;$$

And on account of t being greater than unity, at least one of the c's exceeds 10; and all the c's are less than 10^2, (because the identity is satisfied for $t = 2$.) In the example (i),

$$a_1 = 6,\quad a_2 = 3,\quad a_3 = 8;$$
$$b_1 = 7,\quad b_2 = 2;$$

The identity of my solution gives

$$c_1 = 13,\ c_2 = 48,\ c_3 = 47,\ c_4 = 63,\ c_5 = 16;$$

Here the c's are all greater than 10, and are all less than 10^2.

In the example (ii) also,

$$a_1 = 3,\ a_2 = 1,\ a_3 = 7;\ b_1 = 2,\ b_2 = 1,\ b_3 = 8;$$
$$c_1 = 5,\ c_2 = 9,\ c_3 = 25,\ c_4 = 41,\ c_5 = 30,\ c_6 = 56;$$

Some of the c's exceed 10, and all are less than 10^2.

We can obtain as many number pairs satisfying this second property as we wish

$$910198 \times 9302 = 84\ 66\ 66\ 17\ 96;$$
$$9501 \times 9401 = 89\ 31\ 89\ 01;$$

and so on.

On the other hand, if the identity of my solution does not hold for $x = 10^2$ this second property will be violated. And in case it holds for $x = 10^3$, we get a third property *viz.* of regarding groups of three digits as units between which groups of 9's and 0's are to be inserted. Examples:

$$\left.\begin{array}{l} 991009 \times 991009 = 932\ 098\ 838\ 081 \\ 99910009 \times 99910009 = 9932009898380081 \end{array}\right\} \text{(i)}$$

$$991009992001 \times 991009993001 = 982\ 100\ 805\ 236\ 832\ 075\ 98500,$$

$$\left.\begin{array}{l} 9991000999920001 \times 9991000999930001 \\ \quad = 99820100980502369832007599850001 \end{array}\right\} \text{(ii)}$$

In the first of these two examples,

$$a_1 = 9, \quad a_2 = 9;$$
$$b_1 = 9, \quad b_2 = 9;$$

$c_1 = 18$, $c_2 = 98$, $c_3 = 162$, $c_4 = 81$; Here at least one of the c's exceeded 10^2 and all the c's are less than 10^3.

Therefore we generalise that, when the following conditions hold, we have the property of regarding groups of t digits as units between which groups of nines and zeroes are to be inserted:

Conditions :

(1) $10^t > c_{2r}$, $10^t > c_{2r+1}$, $t \geqslant 2$;

This condition must hold for all the c's.

(2) $c_{2r} \geqslant 10^{t-1}$, $c_{2r+1} > 10^{t-1}$, $t > 2$;

This condition must hold for at least one of the c's.

We may also mention that any number-pair, having in each number more than ten digits, cannot have the first property since some of the c's exceed the limit 10 whatever be the values of the digits a and b.

Question 1765

D. R. KAPREKAR :—If A, B, C, D are four integers each containing the same number of digits such that $(A+C)=(9)^k$ and $(B+D)=(9)^k$ then

$$(A \times B)+(C \times D)+(A \times D)+(B \times C)+A+B+C+D=(9)^{2k}$$

(The symbol $(9)^l$ is the integer formed by the digit 9 repeated l times).

Solution by V. Narasimhamurti, V. Krishnan, B. R. Venkataraman, J. Mahrenholz, B. G. Pendse, H. Z. Gilmann, G. Venkatasubbiah, and the proposer:

$$\begin{aligned} \text{Now } & (A \times B)+(C \times D)+(A \times D)+(B \times C)+A+B+C+D \\ &= (A+C)(B+D)+(A+C)+(B+D) \\ &= (A+C+1)(B+D+1)-1 = 10^{2k}-1 = (9)^{2k} \end{aligned}$$

Further Remarks by B. G. Pendse

It should be noticed that the proof is not affected even if we change A, B, C, D provided A + C and B + D remain the same. Hence the restriction that A, B, C, D should have the same number of

digits may be omitted. Thus let $A=8$, $C=91$; $B=2$, $D=97$ so that $A+C=B+D=99$. We still have

$$(A\times B)+(C\times D)+(A\times D)+(B\times C)+A+B+C+D=9999$$

The following results are generalisations of Question 1765 :

(1) If A, B, C, D are integers such that

$$(A+C)=(9)^{l},\ (B+D)=(9)^{m} \text{ then}$$

$$(A\times B)+(C\times D)+(A\times D)+(B\times C)+(A+B+C+D)=(9)^{l+m}$$

(2) If A, B, C, D are integers such that

$$(A+C)=(p)^{l} \text{ and } (B+D)=(p)^{m}$$

p being any integer of one digit, then

$$(A\times B)+(C\times D)+(A\times D)+(B\times C)+(A+B+C+D)\frac{p}{9}=\frac{p}{9}\ (p)^{l+m}$$

when $p=9$ this reduces to (1)

(3) If A, B, C, D are integers such that $(A+C)=(p)^{l}$ and $(B+D)=(q)^{m}$, p and q being integers with a single digit then

$$(A\times B)+(C\times D)+(A\times D)+(B\times C)+\frac{q}{9}(A+C)+\frac{p}{9}(B+D)=\frac{pq}{9}\times(1)^{l+m}$$

To prove the last result, we note that the left side is

$$(A+C)(B+D)+\frac{q}{9}(A+C)+\frac{p}{9}(B+D)$$

$$=(p)^{l}(q)^{m}+\frac{q}{9}(p)^{l}+\frac{p}{9}(q)^{l}$$

$$=\frac{pq}{81}\left\{(10^{l}-1)(10^{m}-1)\right\}+\frac{pq}{81}\left\{10^{l}-1+10^{m}-1\right\}$$

$$=\frac{pq}{81}(10^{l+m}-1)=\frac{pq}{9}(1)^{l+m}$$

Generalisation by the Proposer

If $(A+C)=(p)^{k}$, $(B+D)=(p)^{k}$ placing a single digit number then $(A\times B)+(C\times D)+(A\times D)+(B\times C)+(A+B+C+D)+(p)^{k}(9-p)^{k}=(p)^{2k}$.

Proof : The left side may be written

$$(A+C)(B+D)+(A+B+C+D)+(p)^{k}(9-p)^{k}$$

$$=(p)^{k}(p)^{k}+2(p)^{k}+(p)^{k}(9-p)^{k}$$

$$=(p)^{k}\left\{p(1)^{k}+2+(9-p)(1)^{k}\right\}=p\frac{(10^{k}-1)}{9}\times(10^{k}+1)$$

$$=p\frac{(10^{2k}-1)}{9}=(p)^{2k}.$$

REVIEWS

L. V. GURJAR : *Theory of Elasticity*, Arya Bhushan Press, pp 108, 1937, Price Rs. Four.

This small brochure of about a hundred pages is devoted mainly to the derivation of the principal differential equations, and other fundamental relations of the Theory of Elasticity, and is based on the lectures delivered by the author to post-graduate students. On the *mathematical* side the treatment appears to be unnecessarily long. If the author had used the notations of the vector and tensor calculus the subject-matter of the book could have been condensed to half the original volume, and the value of the book would have been undoubtedly enhanced. On the *physical* side the book does not go very deep either. It is one thing to give the derivation of the differential equations of the problem, but quite another thing to solve these equations subject to given initial and boundary conditions. About this latter fundamental point there is no reference whatsoever. At least the characteristics of boundary value problems in so far as they relate to Elasticity might have been briefly given. Again there is no mention made of the Theory of Plasticity which is a recent branch of Elasticity fundamentally modifying some of the general concepts of the theory.

We also find it hard to agree with the author that a study of the theory of Elasticity is of *vital significance* to the development of Theoretical Physics, which appears to us to be an over-statement of facts.

In the restricted domain to which the author confines himself he has written a clear and readable book which will surely be useful to post-graduate students.

B. S. MADHAVA RAO

K. R. GUNJIKAR, *An introduction to the calculus. Part I Differential Calculus*, Oxford University Press, pp. 182.

Text-books on calculus usually fall into two distinct types The first type is meant as a first introduction of the subject to somewhat immature students, Graphs and geometric imagery are freely used and full advantage is taken of geometric intuition. The second type is intended for advanced students, and aims at a rigorous treatment of the fundamentals of the subject. There is noticeable in the first type of books a tendency to indulge in loose statements, which may have to be unlearnt at a later stage, and to shift the emphasis from a grasp of the fundamental principles to the attainment of mechanical proficiency in solving problems. In the other type, there is the danger that an undue insistence on rigour to the exclusion of whatever intuition has to tell us, and a disdain of the many practical applications may choke the enthusiasm of the learner, by making the subject a dry and highly introverted science. For the B. Sc. students of Indian universities, neither of these types is quite suited. There is need for striking out a *via media* plan, in which

beginning from ideas suggested by intuition, one is led by gradual steps to a rigorous analytic formulation, and there is a clear statement of what are assumed and what are proved. Professor Gunjikar has brought to bear his long experience in the teaching of calculus, in getting up a book supplying this need.

The limit idea, which is fundamental in calculus is first taken up. The intuitive concepts of the speed of a particle, and the tangent to a curve are analysed, and the way is prepared for the introduction of the differential coefficient. Continuous functions and the various types of discontinuity are studied in chapter III. Infinitesimals, and differentials are carefully treated. Chapter VII is devoted to real numbers, Dedekindian sections and real variable. The exponential and logarithmic functions are introduced, by assuming that the curve $y=10^x$ has a gradient at the point (0, 1). It is refreshing to find the historical note at the end of chapter IV lending a human interest to the subject.

Considerable pains have been taken to expound the theory in a lucid and logical manner, and point out its applications by means of graded and well-chosen sets of examples.

The get-up-excellent. In § 3·21, the symbol l may easily be mistaken for 1, and the difficulty may be avoided by keeping to small l.

It is stated that the text-book is meant as a first course for College students. Rightly enough, the elements of calculus are introduced in the Intermediate classes of most of the Indian universities. It is deplorable that there is no uniformity in the syllabus. Still the book under review seems to be too advanced for the Intermediate for the freshmen if at all it is meant for them. It can be strongly recommended as an excellent text-book for the B. Sc. students of Indian universities. A chapter or at least an article on curvature might have been, with profit, incorporated.

B. RAMAMURTI.

M. BIERNACKI: *Les Fonctions Multivalentes*, (Actualités Scientifiques et Industrielles, No. 657). Price 20 frcs.

This booklet aims at bringing together the results of the recent researches on the Theory of Multivalent Functions. The notion of multivalence is due to P. Mortel who was the first to study systematically the properties of such functions. A function $f(z)$ is said to be *p-valent* in a domain D if it does not take *more than p times* any of its values in D *and* takes *exactly p times* at least one of its values in D. When $f(z)$ takes every one of its values exactly p-times in D it is said to be exactly p-valent. Some allied notions are also considered in the booklet under review, especially that of quasi-multivalence which restricts p-valence to a set of values belonging to a given aggregate, the remaining values being taken more than p-times in D.

When $p=1$, we get univalent functions. These are otherwise known as simple (*schlicht*) functions. When $f(z)$ is regular and simple in a domain D the relation $w=f(z)$ represents D on another domain in a conformal and one-to-one manner. The properties of functions simple in $|z|<1$ has

been studied in great detail. The researches on multivalent functions have been mainly directed to prove and generalise results already known for simple functions. For instance, if $f(z)=a_1 z+a_2 z^2+\cdots$ be simple in $|z|<1$, then $a_n=O(n)$ and $f(z)=O[(1-r)^{-2}]$ as $r=|z|\to 1$. Miss Cartwright has proved that if $f(z)=a_p z^p+a_{p+1}z^{p+1}+\cdots$ be p-valent in $|z|<1$, then $f(z)=O[(1-r)^{-2p}]$ as $r\to 1$. Similarly it has been proved that $a_n=O(n^{2p-1})$ and these are the best possible results.

The Introductory chapter in the booklet under review contains the definitions of p-valence and allied notions, some conditions for p-valence and an account of the relation between quasi-normal families and p valent analytic functions. Chapter I is devoted to a study of multivalent holomorphic functions, Chapter II to meromorphic functions and the last chapter contains some properties of sets of functions in relation to their multivalence.

There is an exhaustive bibliography at the end. As the author himself remarks, the booklet is "intended to collect and group together the results on Multivalent Functions so as to facilitate further researches on the subject"

V. GANAPATHY IYER.

QUESTIONS FOR SOLUTION

1775. (A. A. KRISHNASWAMI AYYANGAR):—Prove the following results:

$$\frac{\pi}{\sqrt{12}}=1-\frac{1}{3\cdot 3}+\frac{1}{5\cdot 3^2}-\frac{1}{7\cdot 3^3}+\cdots$$

$$\frac{\pi}{16}=\frac{1}{1^5+4\cdot 1}-\frac{1}{3^5+4\cdot 3}+\frac{1}{5^5+4\cdot 5}-\cdots$$

$$\frac{\pi-3}{4}=\frac{1}{3^3-3}-\frac{1}{5^3-5}+\frac{1}{7^3-7}-+\cdots$$

$$\frac{4-\pi}{8}=\frac{1}{4^2-1}+\frac{1}{8^2-1}+\frac{1}{12^2-1}+\cdots \qquad (\textit{Tantrasangraha})$$

$$\frac{\pi-3}{6}=\frac{1}{(2\cdot 2^2-1)^2-2^2}+\frac{1}{(2\cdot 4^2-1)^2-4^2}+\frac{1}{(2\cdot 6^2-1)^2-6^2}+\cdots$$

(*Karanapaddhati*, 1730)

[The invention of infinite series of the above forms orginated in Malabar as early as the 15th century. In the *Transactions of the Royal Asiatic Society of Great Britain and Ireland* Vol. III Part III (1834), Charles M. Whish of the Hon. East India Company's Civil Service in the Madras Establishment draws our attention to these series and writes: "The author of *Karanapaddhati* whose grandson is now alive in the 70th year was Pathumana Somayaji, a Nambudri Brahmin of Trichur in Malabar. . . . . The Author of *Tantrasangraha* who was educated in the College of Trichur laid the

foundations of a complete system of fluxions. His name is Tala Kalattura Nambudri of Kerala and he flourished in the 46th Century of *Kaliyuga* (4532 *Kali*—1432 A. D.) . . . . The first series for $\pi/\sqrt{12}$ is exactly similar to that laid down by Dr. Halley and communicated to the *Royal Society of London.*"

Will any historian of Malabar give any further details about these authors and their works?]

1776. (B. R. VENKATARAMAN):—Show that the envelope of lines in a plane π which are equidistant from two fixed points A, B (not lying on π) is a parabola; and that as π revolves about a fixed line l on it, the focus of the parabola describes a circle on PM as diameter in a plane normal to l, where P is the mid-point of AB and M is the foot of the perpendicular from P on l.

1777. (V. THEBAULT, Le Mans, France):—Find a square of three digits abc given that the number of 19 digits $a99999999b00000000c$ is also a perfect square.

1778. (D. R. KAPREKAR):—A number $abcxylmn$ of eight digits divided by a number N of six digits yields a quotient equal to the integral part of $(abc)/3$ with a remainder $1xy999$.

Example: $13313867 \div 299997 = 44 + (113999 \div 299997)$.

Prove that there are only 300 numbers $abcxylmn$ and find the greatest and least of them.

1779. (S. CHOWLA) Is it possible to divide the volume of a rectangular parallelopiped into *unequal* cubes?

(It is known that the area of a rectangle can be divided into nine unequal squares.)

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TIME-MEASUREMENT ANCIENT, MEDIÆVAL AND MODERN

BY

SUKUMAR RANJAN DAS, *Delhi.*

Long before the physical and philosophical notion of Time was developed, it was found necessary to have a standard measurement of Time for all practical purposes—religious as well as secular. There arose in all the ecclesiastical schools of the ancient nations the necessity for instructing some member or group of the priestly order in the process of computing the dates of the religious festivities. [1] The problem was universal and was not confined to any particular religious sect. Since most early religions were connected with sun worship or with astrology, work somewhat similar to that of preparing the computus was needed in all religious organizations—Hindu, Greek, Egyptian, Chinese, Babylonian, Hebrew, Mahammadan and ancient Roman. [2] Among all the nations, the fundamental periods of Time, the day, the month, the year, are the same, the variations occurring in them being principally in the arrangement of the days to form months and years, in the subdivisions of the day; in the times to be reckoned as the commencement of the day, whether at midnight, sunrise, or noon; in the sub-divisions of the year into months, differing from each other as to the number of days of each; in the various kinds of months to form the year, and the like. There have been in all nations certain difficulties experienced as to the time when the year should be reckoned to begin, and in the consequent arrangement of the months and seasons, so that they should recur at regular intervals.

Naturally the revolution of the moon and the sun served for fixing the standard by which Time was to be measured. The ancient thinkers were struck with the daily appearance and disappearance of the moon and the sun. Consequently the movement of the moon or

(1) The Discovery of Time by J. T. Shotwell, *Journal of Philosophy, Psychology and Scientific Methods* XII.

(2) Egyptian papyrus of the beginning of the Christian Era is still extant.

the sun was chosen to fix the measure of Time. The early religion of the ancient nations reveals an intimate knowledge of the times and seasons, and there was from the earliest times an attempt to prepare a calendar setting forth the order in which the rites and ceremonies of the nations should be observed. This calendar, in the earlier periods, was of an imperfect character, which led to methods afterwards adopted for its improvement, generally with a view to its adaptation to religious rather than to secular uses.

The ancient Hindus prepared their calendar mainly for sacrificial purposes, and the performance of various sacrifices facilitated the maintenance of the calendar. When the course of sacrifices was completed, it was found that the year had also run its course, and the sacrifice and the year, therefore, became synonymous terms. There are several sacrificial hymns in the Rigveda (3000 B.C), which show that the sacrificial ceremonies must have been considerably developed; and as no sacrificial system could be developed without the knowledge of months, seasons and the year, it will not be too much to presume that in the Vedic times there must have existed a calendar to regulate the sacrifices. It is difficult to determine the exact nature of this calendar, but a study of the sacrificial literature would show that the phases of the moon, the changes in the seasons, and the southern and northern courses of the sun were the principal landmarks in the measurement of Time in those early days.[3] The difficulties experienced by the Hindus in adjusting their calendar occasioned repeated changes of their system. At one period the motion of the moon was taken as its foundation, and the lunar month was formed to agree with the phases of the moon. The ancient Hindus found that the moon totally disappeared one night and again became full and round another night; they called these phases new and full moon and further observed that from one new moon to another or from one full moon to another the sun rises thirty times. Hence one lunar month became equivalent to thirty days. Then a change in the calendar took place, and a solar month was formed, constituted so as to be reckoned by the time the sun, in its progress, remained in each sign of the solar Zodiac. Another change followed, efforts being made to reconcile the two previous systems, in which each kind of month preserved its original character, the solar month being reckoned in ordinary civil days, and the lunar months measured by lunar days, each being one-thirtieth part of a synodic period, the time elapsing between two conjunctions of the sun and the moon.

(3) Orion by B. G. Tilak, page 11.

The result of these efforts was the formation of the luni-solar year, either in civil days or in lunar days. The Hindus further observed that the star that rose or set at sunrise one day, would not do so after the lapse of several days. They concluded that the sun like the moon moved among the stars in the heavens and that the sun took twelve months to complete this course. Thus according to this calculation a year contains twelve months. The Hindus made several changes regarding the beginning of the day. The Vedic and the Paurāṇic literature maintained that the day began with sunrise: but different theories on this question were advanced by the later Hindu writers. Āryabhaṭṭa maintained that the beginning of the day is to be reckoned from sunrise at Laṅkā. Varāhamihira held that the day begins from midnight. In fact, there is mention of four kinds of day-beginnings, namely, from sunrise, from midnight, from midday and from sunset.[4] The division of the calendar into a year, months, weeks and days evolved gradually but the system was almost complete in the Vedic and the Paurāṇic ages. Slight modifications were introduced later on by the Hindu astronomers and this modified calendar has been since then in vogue in India. The Hindus at a very early date invented the sundial (Gnomon measuring twelve fingers) to measure the sun shadow from which time was determined. The observation of the increase and decrease of the shadow of a tree must have struck them with the idea of a Gnomon. But though the Gnomon was sufficient to measure the time during the day, it was not possible to use it after dusk. The water-clock or clepsydra (a metal bowl floating in a vessel of water where the amount of water that measures a nāḍikā or 24 minutes is given) was, therefore, invented by the ancient Hindus. In fact, they became so skilful in the use of water-clock or clepsydra that they could find out the exact time at a mere look at the instrument. There was also another instrument called Yaṣṭi or staff to measure the time from sunrise or the time after midday.[5]

The Chaldeans knew the length of the year as 365 days 6 hrs. 11 m., but used both the lunar month and the lunar year for civil purposes. They divided both the natural day and the natural night into twelve hours each. They invented the sundial and the water-clock to measure time, the former was used during the day and the latter during the night. For astronomical purposes they divided the day into twenty-four equal hours. They also used very early a fourth of

(4) *Vide* a paper on Hindu Calendar by Sukumar Ranjan Das, *Indian Historical Quarterly*, September, 1928.

(5) *Siddhānta Śiromaṇi*, Golādhyāya, Chapter XI, verses 28-30.

a month (as a convenient division of Time; this was probably half of the half month as was customary in ancient world.[6]

Earlier than 2000 B.C., the Chinese attempted to prepare a calendar. But the system at first changed with each emperor. Under the emperor Yan (c. 2357 B.C.—c. 2258 B.C.) an effort was made to establish a scientific calendar for the whole country, and possibly this was done even earlier, under the emperor Huang-ti (c. 2700 B.C.)[7] There is evidence that according to a decree of Wn-wang (1122 B.C.) the day was arranged to have begun with mid-night, although before this under the Shang dynasty (1766-1122 B.C.) the day began at noon. In the Chinese calendar the civil day has twelve hours and the middle of the first hour is midnight. These hours which were called Shi in the Chinese language are each 120 European minutes in length. Each hour is again divided into eight parts called Khe which is equal to ¼ of an European hour. Each of these parts is then divided into 15 Fen, one fen being equal to one European minute. Each fen is again divided into 60 miao, one miao being equal to one European second. At present also the American clock is becoming common in China. The Chinese days were named in such a way as to give seven-day periods and the month began with new moon.

As early as the 14th century B.C. the Egyptians recognised the value of a fixed year, but the changing one was so strongly implanted in the religious canons of the people that it could not be possibly given up. The fixed year was used to the extent of a division into three seasons, regulated by the river,—the Water Season, the Garden Season, and the Fruit Season (namely, June 21 to October 20; October 21 to February 20; February 21 to June 20). These were easily determined by the Temple observers. They were left with the determination of the seasons and were the principal calendar-makers. From the temple, too, came the announcements of the turn in the rise or fall of the river, the nilometer being under the observation of the temple-priests.[8] In the ancient Egyptian calendar, the business day included the night, the natural day and night being each divided into twelve hours, these hours varying in length with the season. The civil day seems to have commonly begun at sunset. But it is said by Pliny that the priests began their day at midnight. In later times the civil day began at noon, and was divided into twenty-four

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(6) "Resume de Chronologie Astronomique," *Memoire de l' Academie des Sciences*, XXII, pp. 209-476, by J. B. Biot.

(7) History of Mathematics in China and Japan by Mikami, pp. 5-45.

(8) For the details *vide* The Evolution of Calendars by M. B. Cotsworth, Washington, 1922.

equal hours. This was also the case with Ptolemy the astronomer (c. 150 B.C.). In the native Egyptian calendar each month except the last (Mesori) contained thirty days; five days used to be added to Mesori so as to make the year equal to 365 days. This gave an error of ¼ of a day. The year was a changing one and came back to its original position with respect to the heavenly bodies once in 4×365 common years, or 1460 years (1461 Egyptian years). The year began with the first day of month Thoth, the God, who was supposed to introduce the calendar and numbers into Egypt.

After Egypt became a Roman province (c. 30 B.C.) the Alexandrian calendar, including the fixed year. was introduced, although the varying year remained until the fourth century A.D. The Alexandrian system was used in the first half of the seventh century till 638 A.D. when Alexandria yielded to Mahamadan conquest. There was then a change in the calendar except in Upper Egypt. In latter days when the French obtained brief control of Egypt in 1798 the European system was used in Egypt side by side with the Mahammadan system.

The Athenian calendar[9] followed the Egyptian in beginning the new day at sunset and in dividing both day and night into twelve hours. The seven-day week was not used. However, the lunar month was divided into three parts. The first part was equal to ten days numbered in order, the " fifth day of the beginning of the month " being the fifth. Then followed nine days, numbered as before, but with the designation "over ten ", i.e., "one over ten " and so on to "twenty ". Then to the end of the month the numbers were " one over twenty " and so on. These days were also used to be numbered backwards from the end of the month. In the popular calendar the month began with the new moon, and twelve of these months made 354 days, requiring the insertion of a new month every three years. This was called the second month of Poseideon, known as Poseideon II. In 432 B.C. Meton constructed a nineteen-year cycle in which the third, fifth, eighth, eleventh, thirteenth, sixteenth and nineteenth years used to contain the extra month. Nineteen years contained two hundred and thirty-five months and were equal to $6939\frac{3}{4}$ days. These months as arranged, however, contained $6940\frac{5}{9}$ days. In 325 B.C. Callippus modified it to include four nineteen-year cycles. 4×19 years = 76 years = 940 months. The months were 29 or 30 days and therefore 940 months were equal to 27759 days. Still

(9) History of Mathematics, Vol. II by D. E. Smith, pp. 660-664.

later in 150 B.C. Hipparchus suggested the use of four of the cycles of Callippus. But none of the last two calendars ever came into popular use.

The Romans considered the planets as ruling over one hour of each day, in the following order, beginning with the first hour of Saturday: Saturn, Jupiter, Mars, Sun, Venus, Mercury, Moon. At that time the sun and moon were placed among the wanderers (planets). Taking Saturn for the first hour of Saturday and counting the hours forward we find that the second hour is ruled by Jupiter and so on to the twenty-fourth, which is ruled by Mars. Then the next hour, the first of Sunday, is ruled by the Sun, the first of the next day by the Moon and so on. Thus the days of the week were named after the ruling planets of their first hours. Hence we have Saturn's day, Sun's day, Moon's day, Mar's day (French Mardi) Mercury's day (French Mercredi), Jupiter's day (In the northern lands, Thor's day), Venus's day (Frigg's day, Frigg was the goddess of marriage). The oldest of the Roman calendars is attributed to Romulus, the founder of Rome. The year, then, probably consisted of ten months of varying lengths, or of 304 days. The year began with March. Numa Pompilius (715-672 B.C.) added two other months, January and February, and his year was probably lunar. In the fifth century B. C. the Decemvirs decreed a solar year, the regulation of which was left in the hands of the priests. The calendar was so mismanaged that by the time of Julius Caesar each day was eighty days out of its astronomical place. Radical measures were, therefore, necessary for the reform of the calendar and Caesar decreed that the year 46 B C. should have 445 days and that thereafter the year should consist of 365 days with a leap year every fourth year. The difficulty arose thus; the ordinary civil year contains an exact number of days, viz. 365, but the time taken by the *sun to complete a revolution in the ecliptic is about 365¼ days and* the exact interval between the successive vernal equinoxes is 365 days 5 hrs. 48 min. 45·5 sec. which is the Tropical Year. Therefore, by taking the civil year as 365 days, there is an error compared with the tropical year of 5 h. 48 m. 45·5 sec., which in four years amounts to 23 h. 15 m. 2 sec., or very nearly a day. If this error was not corrected, the result would be that the dates of the equinoxes and solstices would be later by one day every four years. The first exact attempt at approximating the length of the civil to that of the tropical year was made by Julius Caesar. It was then agreed that an additional day should be given to every fourth year, which was to contain 366 days. Caesar followed the following plan

to determine the names of the months and the number of days to constitute each month:—

Names of the months					Number of days
1.	Martius	...	...	...	31
2.	Aprilio	...	...	...	30
3.	Maius	...	...	...	31
4.	Junius	...	...	...	30
5.	Quintilis	...	...	...	31
6.	Sextilis	...	...	...	31
7.	Septembris	...	...	...	30
8.	Octobris	...	...	...	31
9.	Novembris	...	...	...	30
10.	Decembris	...	...	...	31
11.	Januarius	...	...	...	31
12.	Februarius	...	...	...	28

Here the year was made to begin in March. An account is also found for the origin of the nomenclature of September (seventh month), October, November, and December. In his original plan Caesar caused every alternate month, beginning with March, to have 31 days, the others having 30 days, except that February received its 30th day once in four years, otherwise it had 29 days. Later Julius Caesar decreed that the year should begin with January. Finally, during his life-time he changed the name of Quintilis, the month in which he was born, to that of Julius, after his own name. He also changed the number of days in certain months and the result was the present Julian Calendar. After the death of Julius Caesar and in the second year of his calendar a further confusion arose, apparently through a misunderstanding on the part of the priests as regards the proper date for leap year. This was corrected by Augustus, and in his honour the name of Sextilis was changed to bear his name. Since then the Julian calendar was in vogue till it was further reformed by Pope Gregory XIII in 1582. According to the Julian Calendar a correction was made of one day in four years. But one day, or 24 hours, was in excess of 23 hrs. 15 min. 2 sec. by about 45 minutes. Thus the correction by means of leap year led to a new but very much smaller error of about 45 minutes in four years, or an average of rather more than eleven minutes each year. This error, in 400 years, would amount to nearly three days. Hence was necessary the Gregorian correction to the Julian Calendar adopted by Pope Gregory, according to which each year which is a multiple of 100, such as 1700, 1800, 1900, which by the Julian Calendar, are

leap years, should be ordinary years, with the exception of those years in which the number of the century is divisible by 4 without remainder, such as 2000, 2400, which should remain leap years. This arrangement evidently makes the required correction of three days in 400 years. Even with the Gregorian correction there is still a very small error, which, however, would amount to not more than a day in 20,000 years. The Gregorian correction was not adopted in England until the year 1752, when the accumulated error, as compared with the corrected calendar, amounted to eleven days. Eleven days of 1752 were, therefore, skipped, the 2nd of September being called the 13th. The Julian Calendar was used by the Greek Catholics, including the Russians until the world war of 1914-1918, the dates at that time differred by thirteen days from those of the calendar of Western Europe where the Gregorian correction was adopted.

It is interesting to note that in the early centuries the year in the Christian calendar usually began with April in the East of Europe[10] and with March in the West, although sometimes with the Feast of the Conception, Christmas day, Easter, or Ascension day, or at other times according to the orders of the Popes. In Spain until the sixteenth century and in Germany from the 11th century, March 1 and March 25 were the favourite days to make the year-beginning, although Advent Sunday (the fourth Sunday before Christmas) has generally been recognised as the beginning of the ecclesiastical year. March 1 was generally used in mediaeval France for beginning the year. The same was the custom in Oriental Christendom, and in Venice until 1797. March 25 was used by the mediaeval Pisans and Florentines for year-beginning. In Italy Pope Innocent XII decreed that the year should begin on January 1, beginning with 1691, as Philip II had done for the Netherlands in 1575, and as Julius Caesar had done before the Christian Era. Most of the Italian states adopted January 1 in 1750. England adopted it in 1752. [11]

We may mention here that there is evidence of several changes of the year-beginning in the Hindu calendar. In the early Vedic times the year began when the sun was in the vernal equinox. Later on, the commencement of the year was changed from the vernal equinox to the winter solstice. It is difficult to ascertain definitely the time of the change. Now to understand this change

(10) The Byzantine Calendar began with September 1.

(11) History of Mathematics, Vol. II page 651 and pages 661-670, by D. E. Smith.

in the beginning of the year, it is necessary to remember that the solar year was sidereal and not tropical in the case of the Hindu Calendar, and that the great object of the calendar was to ascertain the proper time of the seasons. This necessitated a change in the beginning of the year, every two thousand years or so, to make it correspond with the cycle of natural seasons. The difference between the sidereal and the tropical year is 20·4 minutes, which causes the seasons to fall back nearly one lunar month in about two thousand years, if the sidereal solar year be taken as th› standard of measurement. Therefore, the beginning of the year was twice altered owing to the procession of the equinoxes. The third change in the year-beginning was introduced at the time of the Vedanga Jyotisa (about 500 B.C.), when the seasons had further receded by a fortnight and the beginning of the month was altered from the full-moon to the new-moon during this period. The next change was introduced and put into effect by the astronomer Varahamihira, in the beginning of the sixth century A. D. and this last system is even now being used in India. [12]

The Jews began their day at sunset, their week on Saturday night (i.e., when their holyday ends and Sunday begins) and their year with Fishi L (the first new moon after the autumnal equinox).[13] Their calendar was lunar. In the ancient Maya civilization the year began with the Winter Solstice, was divided into eighteen months and was entirely independent of astronomical considerations. Scholars assert that their calendar goes back to the thirty-fourth century B.C. The Mahammadans began their day with sunset, divided both day time and night time into twelve hours, the length of the hour varying with the season; their week began on Sunday, their month with the new moon, and the year was purely lunar of 354 or 355 days.

When the year-beginning was fixed it was found necessary to determine the date from which the years were to be numbered. In India the Saka Era which is 78 years behind the Christian Era is generally followed, i.e., the date of accession of a famous Saka King is taken to number the years. In Europe, following the Roman custom, the years in the early centuries were dated from the accession of the emperor or consul until the abbot Dionysius Exignus (533 A.D) arranged the Christian calendar in such a way

(12) Seasons and Year-beginnings of the Hindus by Sukumar Ranjan Das.

(13) Elements of the Jewish and Mahammadan Calendars, by S. B. Bunaby, London 1901.

as the supposed date of the birth of Christ was generally taken for the beginning of the era. This era was adopted in Rome in the sixth century. The Mahammadans begin their era from the date of birth of their Prophet, Mahammad. To get the Christian year from the hejira (Mahammadan year), we add 622 to 97 per cent of the number of the hejira year, i.e., 1300 A. H $= \frac{97}{100} \times 1300 + 622 = 1883$ A.D.

In the early days of the French revolution an attempt was made to impose a new calendar upon the country, partly as a protest against the Christian Church. The calendar was to begin with the autumnal equinox on September 22, 1792. The months of this calendar were named according to natural conditions.

Of the chief divisions of time the most obvious one was the day. This was, therefore, the primitive unit in the measurement of time and the one which for many generations must have been looked upon as unvarying. As the human race developed, however. various kinds of day were distinguished. First from the standpoint of invariability is the sidereal day, the interval of time taken by the fixed stars to complete a revolution round the pole, namely 23 hrs. 56 m. 4·09 seconds of our common time. But from the standpoint of the casual observer, however, first is the true solar day, the length of time between one passage of the sun's centre across the meridian and the next passage. This latter day varies with the season, the difference between the longest and shortest day being 51 seconds; but for common purposes the solar day sufficed for thousands of years and the sundial was frequently used to measure the length of time. As clocks became perfected a third kind of day came into use, the artificial mean solar day which is the average of the variable solar days of the year and equal to 24 hrs. 3 m. 56·56 seconds of sidereal time. In addition to these general and obvious kinds of day there are others which have been mentioned by the writers on chronology. With the Babylonians the day began at sunrise; with the ancient Hindus it began at sunrise or midday or midnight, or sunset at various periods, but most generally from sunrise; with the Athenians, Jews and various other ancient peoples and with certain Christian sects at sunset; with the Umbrians at noon or midday; and with the Roman and Egyptian priests at midnight.[14]

The next obvious division of time was the month; it was originally the length of time from one new-moon or one full-moon to the next. This was generally used by the ancient nations and served as

(14) Ancient Calendars and Cycles by Plumett.

the greater unit for many thousands of years. However, as science made considerable progress, it became apparent, as in the case of the day, that there are several kinds of months. There is a sidereal month, the time required for a passage of the moon about the earth as observed with reference to the fixed stars, namely, 27 days 7 hours 43 minutes 11·5 sec. There is also the synodic month, the interval of time from one conjunction of the sun and the moon to the next one, the average length of which is 29 days 12 hours 44 min. 3 sec. or two days 5 hours 0 min. 51·5 sec. more than the sidereal month. This is the month used by those among whom a lunar calendar is in vogue. It is the basis of the artificial month twelve of which make our common year.

Less obvious than the day or the month is the year, both sidereal and tropical. A sidereal year is the time taken by the sun to return to the same position relative to the fixed stars and a tropical year is the exact interval between two successive vernal equinoxes. But the ordinary civil year which is in use, contains an exact number of days, viz. 365 and the time taken by the sun to complete a revolution in the ecliptic is about 365¼ days. Probably at first the year recognised by the primitive nations was the lunar year consisting of twelve synodical months and is even now used in some parts of the world, specially by the Mahammadans.

The least obvious division of time was the week. It seems very likely that it arose from the need for a longer period than the day and a shorter than the month. The Chaldeans were the first who named the seven days of the week after the respective presiding planets and it is believed that the Hindus got this practice of naming the seven days from the Chaldeans.[15] The Romans also named the days of the week after the supposed presiding planets. This division of the week into seven days is in vogue throughout the world. But recently in Soviet Russia there has been an attempt to make the week contain five days, to be named after the five principal planets excluding the sun and the moon. Probably the week was conceived to be made up of seven days as according to the ancient nations there were seven planets including the sun and the moon.

In all the ancient calendars there were usually twelve hours in the day and twelve hours in the night. It is still a mystery why the number twelve was chosen. Some suggest that the reason is found in the custom relating to the Babylonian knowledge of the inscribed hexagon. Some again say that this was selected because from twelve

(15) Indian Astronomy (in Marathi language) by S. B. Dikshit p. 138 f.

the fractions ½, ⅓, ¼ could easily be obtained. The day hours are longer than the night hours in the summer, and shorter in the winter. This fact is referred to by several ancient writers.[16]

To find the hours of the day the shadow cast by some obstruction to the sun's rays was used. Then an artificial gnomon was erected and lines were drawn on the earth to mark off the shadows. Since the hour-shadow is longer when the sun is near the horizon, either concave surfaces or curved lines on a plane were placed at the foot of the gnomon. The sundial was first used in Babylonia, in India and in Egypt. From Babylonia it came to Greece where it was introduced by Anaximander (c. 575 B.C.). In the sundial of that period the gnomon was placed at the centre of three concentric circles. But very often there were difficulties with the gnomon. At times when the sky was cloudy the gnomon did not cast a distinct shadow. Hence the need was early felt for some kind of device to tell the hours at night as well as during the day in cloudy weather and also when the sun's rays were direct. Various methods were employed by the early nations, such as burning of tapers, hour-glasses and water-clocks (or clepsydra). The hour-glass was known as early as 250 B.C. Plato gave much thought to the subject, and his conclusions might have suggested to Ctesibius (c. 150 B.C.) the idea of a water-clock. The Clepsydra was of earlier origin. There is mention of this instrument in the Vedângo Jhotiṣa (500 B.C.). The Clepsydra was introduced into Rome in 159 B.C. in the early form water trickled from one receptacle to another in a given time, as the sand in an hour-glass. It is till recently in operation in one of the ancient towers of Canton in China All these instruments led to the invention of the clock. In the middle ages Boethius (480 A.D. to 525 A.D.) invented a clock (c. 510 A.D.) and it was used in churches as early as 612 A D. The invention of clocks driven by weights is ascribed to Pacificus, archdeacon of Verona, in the 9th century A.D. The pendulum clock was introduced about 1657 A D. and was chiefly due to Huygens.

An attempt for the measurement of time and to fix the hour of the day as well as to prepare a calendar for practical purposes was as ancient as the human race. The evolution has, no doubt, been gradual, but the introduction of time in chronology dates to a prehistoric period and it has passed through various stages from the ancient down to the modern age.

([16]) History of Mathematics, Vol. II, pages 668-670, by D. E. Smith.

SOME PROPERTIES OF BINOMIAL COEFFICIENTS

By

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§ 1·1. The following properties of binomial coefficients are easily proved by operational methods.* We shall use the symbol E defined by E $f(n) = f(n+1)$ which obeys the fundamental laws of algebra and for which, therefore, the expansions and identities hold good.

§ 1·2. Let $F(n)$ denote the expression

$$f(0)+\binom{n}{1}f(1)+\binom{n}{2}f(2)+\cdots+\binom{n}{t}f(t)+\cdots+\binom{n}{n}f(n).^{\dagger}$$

Then it can be shown that

$$\frac{f(0)}{r}+\frac{nf(1)}{r(r+1)}+\frac{n(n-1)}{r(r+1)(r+2)}f(2)+\cdots+\frac{n!\,f(n)}{r(r+1)\cdots(r+n)}=$$

$$\frac{F(0)}{n+r}+\frac{nF(1)}{(n+r)(n+r-1)}+\frac{n(n-1)\,F(2)}{(n+r)(n+r-1)(n+r-2)}+\cdots$$
$$+\frac{n!\,F(n)}{(n+r)(n+r-1)\cdots r}\quad\ldots\quad(1)$$

If r be an integer, by expanding $\dfrac{(r-1)!\,n!}{(n+r)!}\,\dfrac{(1+t)^{n+r}}{1-tE}\,f(0)$ in two ways and equating the coefficients of t^n we get formula (1). If we multiply throughout by $r(r+1)\cdots(r+n)$, both sides become expressions of the $(n+1)$th degree in r. Therefore, as the two sides are equal for all positive integral values of r, (1) is in identity. [The same reasoning may be employed for the following results also].

§ 1·3. Let $f(n)$ be x^n. Then $F(n)$ is equal to $(1+x)^n$.

And so, by (1)

$$\frac{1}{r}+\frac{nx}{r(r+1)}+\frac{n(n-1)x^2}{r(r+1)(r+2)}+\cdots+\frac{n!\,x^n}{r(r+1)\cdots(r+n)}=$$

$$\frac{1}{n+r}+\frac{n(1+x)}{(n+r)(n+r-1)}+\frac{n(n-1)(1+x)^2}{(n+r)(n+r-1)(n+r-2)}+\cdots+\frac{n!\,(1+x)^n}{(n+r)\cdots r}.$$
$$\ldots\quad(2)$$

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* Proofs by induction can be given but they are long and tedious.

† $\binom{n}{r}$ denotes the binomial coefficient nCr.

Differentiating both sides p times and equating coefficients of x^{q-p} we get

$$\frac{(n,q)(q,p)}{r(r+1)\cdots(r+q)}=\frac{(n,q)(q,p)}{(n+r,q+1)}+\frac{(n,q+1)(q+1,p)}{(n+r,q+2)}\cdot\binom{q-p+1}{q-p}+\cdots$$
$$+\frac{(n,q+s)(q+s,p)}{(n+r,q+s+1)}\cdot\binom{q-p+s}{q-p}+\cdots$$
$$+\frac{n!\,(n,p)}{(n+r,n+1)}\cdot\binom{n-p}{q-p},$$

where $(a,\ p)$ denotes $a(a-1)\cdots(a-p+1)$. Changing n into $n+q$ and reducing we get

$$(n+q+r,n)\cdot q!=(r,n)\cdot q!+\binom{n}{1}(r,n-1)\cdot(q+1)!+\cdots$$
$$+\binom{n}{s}(r,n-s)\cdot(q+s)!+\cdots+(q+n)! \qquad \ldots \quad (3)$$

§ 2. $$\frac{(p+q)!}{q!\,x(x+1)\cdots(x+p+q)}=\frac{1}{x(x+1)\cdots(x+q)}$$
$$-\frac{\binom{p}{1}}{(x+1)\cdots(x+q+1)}+\cdots+(-1)^r\frac{\binom{p}{r}}{(x+r)(x+r+1)\cdots(x+r+q)}$$
$$+\cdots+(-1)^p\frac{1}{(x+p)\cdots(x+p+q)} \qquad \ldots \quad (4)$$

For, if we denote $\frac{(-1)^s}{x+s}$ by $f(s)$, and $\frac{p!}{x(r+1)\cdots(x+p)}$ by $F(p)$,

$$F(p+q)=\frac{(p+q)!}{x(x+1)\cdots(x+p+q)}=(1+E)^{p+q}f(0).$$

The right side, multiplied by $q!$, is equal to

$$(1+E)^q f(0)+\binom{p}{1}(1+E)^q f(1)+\cdots+\binom{p}{p}(1+E)^q f(p)$$
$$=(1+E)^q(1+E)^p f(0).$$

Thus we get the required result.

§ 3. If $\phi(n)$ denotes $f(0)+\binom{n}{1}f(1)+\binom{n+1}{2}f(2)+\binom{n+2}{3}f(3)+\cdots$, then

$$\frac{\phi(1)}{n+r}+\frac{n\phi(2)}{(n+r)(n+r-1)}+\frac{n(n-1)\,\phi(3)}{(n+r)(n+r-1)(n+r-2)}+\cdots+\frac{n!\,\phi(n+1)}{(n+r)\cdots r}$$
$$=\frac{f(0)}{r}+\frac{n+r+1}{r(r+1)}f(1)+\frac{(n+r+1)(n+r+2)}{r(r+1)(r+2)}f(2)+\cdots\cdots \qquad \ldots \quad (5)$$

By expanding $\frac{n!\,(r-1)!}{(n+r)!}\;\frac{(1+t)^{n+r}}{1-E(1+t)}\,f(0)$ in two ways and equating coefficients of t^n we get the required result.

§ 4. With the same notation as in § 3,

$$\frac{\phi(n+1)}{f(0)}=\frac{\frac{f(0)}{r}+\frac{n+r+1}{r(r+1)}f(1)+\frac{(n+r+1)(n+r+2)}{r(r+1)(r+2)}f(2)+\cdots\cdots}{\frac{f(0)}{r}-\binom{n}{1}\frac{f(1)}{r+1}+\binom{n}{2}\frac{f(2)}{r+2}-\cdots\cdots} \quad \ldots \quad (6)$$

It can readily be shown that

$$(1-E)^{n+1}\left\{\frac{1}{r}-\binom{n}{1}\frac{E}{r+1}+\binom{n}{2}\frac{E^2}{r+2}-\cdots\cdots\right\}=$$

$$\frac{1}{r}+\frac{n+r+1}{r(r+1)}E+\frac{(n+r+1)(n+r+2)}{r(r+1)(r+2)}E^2+\cdots\cdots$$

Operating on $f(0)$ by both sides of the above we get the required result.

§ 5. If $\phi(r,s)=f(s)+rf(s+1)+\frac{r(r-1)}{2!}f(s+2)+\cdots\cdots$ then

$$\frac{\phi(n,0)}{2n+1}+\frac{n\cdot\phi(n-1,0)}{(2n+1)\cdot 2n}+\frac{n(n-1)\,\phi(n-2,0)}{(2n+1)\cdot 2n\cdot(2n-1)}+\cdots+\frac{n!\,\phi(0,0)}{(2n+1)\cdot 2n\cdots(n+1)}$$

$$=\frac{\phi(n+1,0)}{n+1}+\frac{2n+2}{(n+1)(n+2)}\phi(n+1,1)$$

$$+\frac{(2n+2)(2n+3)}{(n+1)(n+2)(n+3)}\phi(n+1,2)+\cdots\cdots \quad \ldots \quad (7)$$

This may be proved by expanding $\frac{(n!)^2}{(2n+1)!}\;\frac{(1+t)^{2n+1}(1+E)^{n+1}f(0)}{1-E(1+t)}$ in two ways and equating the coefficients of t^n.

§ 6. If $\psi(n)=f(0)+f(1)+\cdots+f(n)$, then

$$(r+n,n)f(0)+(n,1)(r+n-1,n-1)f(1)+\cdots+(n,s)(r+n-s,n-s)f(s)+\cdots$$

$$=(r+n-1,n)\,\psi(0)+(n,1)(r+n-2,n-1)\,\psi(1)+\cdots\cdots$$

$$+(n,s)(r+n-s-1,n-s)\,\psi(s)+\cdots\cdots \quad \ldots \quad (8)$$

where (a,p) denotes, as in § 1·3, $a(a-1)\cdots\cdots(a-p+1)$.

Since $\psi(n)=\frac{1-E^{n+1}}{1-E}f(0)$, the right side of (8)

$=n!\times$ coefficient of t^n in

$$(1+t)^{n+r-1}\left\{1+\frac{t}{1+t}\cdot\frac{1-E^2}{1-E}+\frac{t^2}{(1+t)^2}\cdot\frac{1-E^3}{1-E}+\cdots\cdots\right\}f(0)$$

$= n! \times$ coefficient of t^n in

$$\frac{(1+t)^{n+r}}{t}\left\{\frac{t}{1+t}(1-E)+\frac{t^2}{(1+t)^2}(1-E^2)+\cdots\right\}\frac{1}{1-E}$$

$= n! \times$ coefficient of t^n in

$$\frac{(1+t)^{n+r}}{t}\left\{\frac{1}{1-\frac{t}{1+t}}-\frac{1}{1-\frac{tE}{1+t}}\right\}\frac{1}{1-E}$$

$$= n! \times \text{coefficient of } t^n \text{ in } \frac{(1+t)^{n+r+1}}{t}\left\{\frac{1+t(1-E)-1}{(1+t-tE)(1-E)}\right\}$$

$$= \text{coefficient of } \quad (1+t)^{n+r}\left\{1-\frac{tE}{1+t}\right\}^{-1}$$

$= n! \times$ coefficient of t^n in

$$\{(1+t)^{n+r}+t(1+t)^{n+r-1}E+t^2(1+t)^{n+r-2}E^2+\cdots\}f(0)$$

= the left side.

§ 6·2. If $f(r)$ is taken to be $\dfrac{x(x+1)\cdots(x+r-1)}{r!}$ and $f(0)$ to be 1, in (8), we have

$$F(n)=1+\frac{x}{1!}+\frac{x(x+1)}{2!}+\cdots+\frac{x(x+1)\cdots(x+n-1)}{n!}$$

$$=\frac{(x+1)(x+2)\cdots(x+n)}{n!},$$

and so

$$(r+n,\,n)+\binom{n}{1}(r+n-1,\,n-1)x+\binom{n}{2}(r+n-2,\,n-2)x(x+1)+\cdots$$

$$=(r+n-1,\,n)+\binom{n}{1}(r+n-2,\,n-1)(x+1)$$

$$+\binom{n}{2}(r+n-3,\,n-2)(x+1)(x+2)+\cdots \quad \ldots \quad (9)$$

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ON THE THREE CONICOIDS OF A TANGENTIAL SYSTEM WHICH PASS THROUGH A POINT

BY

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1. If $\sum = 0$ and $\sum' = 0$ be the tangential equations of two given conicoids, $\sum + \lambda \sum' = 0$ represents a conicoid touching the common tangent planes to the two conicoids i.e., inscribed in the developable common to the two conicoids. Transforming to point coordinates, it will be seen that the equation is a cubic in λ and thus, in general, there will pass through any point in space three members of the system. The discriminant of the above cubic equated to zero gives a surface of the eighth degree on one side of which—we shall call this side the inside—the three roots are all real and on the other side only one root is real. The object of the present paper is to discuss the nature of the three conicoids, when real, through a point.

2. If the common self-conjugate tetrahedron of the two conicoids be taken as the reference tetrahedron, the equation to the system of conicoids in quadriplanar coordinates will be $\sum \dfrac{x^2}{a - \lambda a'} = 0$ where $\sum x^2/a = 0$ and $\sum x^2/a' = 0$ are the equations to the two given conicoids. A conicoid of the system is a hyperboloid of one sheet if two of the coefficients a, b, c, d are positive and the other two negative whereas for an ellipsoid or a two-sheeted hyperboloid, three of the coefficients are of the same sign (say positive) and the fourth negative. To distinguish now between an ellipsoid and a two-sheeted hyperboloid, we consider the section by the plane at infinity which is real for a hyperboloid and imaginary for an ellipsoid. The equation $\sum x^2/a = 0$ with $a, b, c > 0$ and $d < 0$ will represent an ellipsoid if $a+b+c+d<0$ and a hyperboloid of two sheets if $a+b+c+d>0$. The condition for a paraboloid is plainly $a+b+c+d=0$.

3. In what follows two conicoids of the system will be called like when they are conicoids of the same kind with corresponding coefficients of like sign. Clearly we can start with two like conicoids. For the signs of $a - \lambda a'$, $b - \lambda b'$ etc. are the same as, or opposite to those of a', b' etc. if λ is less than the least or greater

than the greatest of a/a', b/b', c/c', d/d'. And as

$$\sum a - \lambda \sum a' = \sum a'(\sum a \, \sum a' - \lambda),$$

we take $\lambda < \sum a / \sum a' = \lambda_0$ if λ_0 is less than the least of a/a', b/b', c/c', d/d' or $\lambda >$ the greatest of λ_0, a/a', b/b', c/c', d/d' if λ_0 is greater than one or more of a/a', b/b', c/c', d/d'.

We suppose that the two given conicoids are not both paraboloids for in that case all the members of the system will also be paraboloids. It will be easily seen that the two conicoids to start with can always be taken to be two like ellipsoids except in one case only. If S and S′ are two like hyperboloids of one sheet we choose λ_1, λ_2 greater than three and less than the fourth of the ratios a/a', b/b', c/c', d/d' so that the equations $\sum x^2/(a-\lambda_1 a') = 0$ and $\sum x^2/(a-\lambda_2 a') = 0$ will represent both like ellipsoids or hyperboloids of two sheets. Taking therefore S and S′ to be like hyperboloids of two sheets with a, b, c; a', b', $c' > 0$ and d, $d' < 0$ we may suppose d/d' to be smaller than all or greater than only one of a/a', b/b', c/c' (otherwise we take $\lambda = 1/\mu$). If d/d' were the smallest, $\lambda_0 < d/d'$ and we choose λ_1, λ_2 so as to lie between d/d' and λ_0. If $a/a' > b/b' > d/d' > c/c$ there is no upper or lower bound for λ_0. In this case we can easily get λ_1, λ_2 so that $\Sigma x^2/(a-\lambda_1 a') = 0$ and $\Sigma x^2/(a-\lambda_2 a')c' = 0$ represent like ellipsoids except when S and S′ are hyperboloids of two sheets and

$$a/a' > \lambda_0 > b/b' > d/d' > c/c'.$$

4. Let then $S \equiv \Sigma x^2/a = 0$ and $S' \equiv \Sigma x^2/a' = 0$ be two like ellipsoids. Let us first suppose $a/a' > b/b' > c/c' > d/d'$ where a, b, c; a', b', $c' > 0$ and d, $d' < 0$. Then $d/d' > \lambda_0$. The signs of $f(\lambda) \equiv \Sigma(b-\lambda b')(c-\lambda c')(d-\lambda d')x^2$ for values $-\infty$, d/d', c/c', b/b', a/a', $+\infty$ of λ are those of $-S'$, $+$, $+$, $-$, $+$, $+S'$. The roots of $f(\lambda)=0$ are therefore all real and the developable is imaginary in this case. One root lies in $(c/c', b/b')$ and another in $(b/b', a/a')$. The third root lies in $(-\infty, d/d')$ or $(a/a', +\infty)$ according as $S' \gtrless 0$ i.e., according as the point lies outside or inside S′. The root which lies in $(c/c', b/b')$ determines a two-sheeted hyperboloid and that in $(b/b', a/a')$ a hyperboloid of one sheet. Inside S′, the third root gives an ellipsoid. Outside S′, it lies in $(d/d', -\infty)$. It therefore gives an ellipsoid or a two-sheeted hyperboloid according as $\lambda \lessgtr \lambda_0$ i.e., according as $f(\lambda_0) \gtrless 0$. Now $f(\lambda_0)=0$ is the equation of the unique paraboloid P of the system and thus in this case "The three conicoids through a point are always real one a hyperboloid of one sheet, the other of two sheets and the third an ellipsoid or a two-sheeted hyperboloid

according as the point lies inside or outside the unique paraboloid P of the system."

5. Let us next suppose $a/a' > b/b' > d/d' > c/c'$. The signs of $f(\lambda)$ for values $-\infty$, c/c', d/d', b/b', a/a', ∞ of λ are those of $-S'$, $-$, $-$, $-$, $+$, S'. Within S' the roots are real lying in $(-\infty, c/c')$, $(b/b', a/a')$, $(a/a', \infty)$. The middle root which is always real gives a one-sheeted hyperboloid. If $\lambda_0 < c/c'$, the paraboloid $f(\lambda_0)=0$ is elliptic and within it $(f(\lambda_0)>0)$ the three conicoids are an ellipsoid, an hyperboloid of one sheet and one of two sheets. If $\lambda_0 > a/a'$, the conicoids are again an ellipsoid, an hyperboloid of two and one sheets for points within the paraboloid $f(\lambda_0)=0$* which is again an elliptic paraboloid.

If λ_0 lies in $(c/c', a/a')$ the two extreme roots for $S'<0$ are both ellipsoids and this region corresponds to $f(\lambda_0)<0$ i.e., outside the paraboloid $f(\lambda_0)=0$. Outside, the signs are $-$, $-$, $-$, $-$, $+$, $+$. Inside P $(f(\lambda_0)>0)$ the other two roots are real if λ_0 does not lie between b/b', a/a'. Supposing $c/c'<\lambda_0<d/d'$, the paraboloid P is hyperbolic and within it the two roots give both hyperboloids of one sheet, whereas if $d/d'<\lambda_0<b/b'$, the two roots for points within P give one an ellipsoid and the other a two-sheeted hyperboloid. The paraboloid P is elliptic in this case.

We have yet to examine the region outside S' and outside P i.e. $S'>0$ and $f(\lambda_0)<0$ (except when $\lambda>a/a'$). Writing $\lambda=\mu+\lambda_0$ we get

$$f(\lambda)\equiv f(\mu+\lambda_0)\equiv(-a'b'c'd')\mu^3+\mu^2T+\mu T+f(\lambda_0)=0$$

has always one root greater than λ_0 ($<\lambda_0$ if $\lambda_0>a/a'$). Hence the other two roots whenever real are both less or both greater than λ_0. Hence if $T>0$ the two roots when real are both $<\lambda_0$ and if $T<0$ and $T'<0$ both are again $<\lambda_0$. But the nature of the corresponding conicoids cannot be determined unless the particular interval in which the roots lie is found out.

6. There is one case yet to be considered viz. when the conicoids S and S' are both like ellipsoids (or 2 sheeted hyperboloids) and $a/a'>\lambda_0>b/b'>d/d'>c/c'$. The signs of $f(\lambda)$ for $-\infty$, c/c', d/d', b/b', a/a', ∞ are those of $-S'$, $-$, $-$, $-$, $+$, $+S'$. One root is always real, lies in $(b/b', a/a')$ and determines a hyperboloid of one sheet. Within S', the other two roots are real and determine two ellipsoids (or 2 sheeted hyperboloids). Within the developable the other two roots are real and they are both less or both greater than λ_0. But

* As three of $a-\lambda a'$, $b-\lambda b'$ etc. are negative.

as in the preceding section unless the particular interval in which the roots lie is determined it is not possible to determine the nature of the two conicoids. All that can be said is that they are both ellipsoids or both hyperboloids of one or two sheets. Hence summing up we have "*of the three conicoids through a point one is always real and is a hyperboloid of one sheet. Within the unique paraboloid P of the system—whenever it is elliptic, the other two conicoids are an ellipsoid and a two-sheeted hyperboloid, whereas if P is hyperbolic the three conicoids are all one sheeted hyperboloid or one a hyperboloid of one sheet and the other two both ellipsoids or two sheeted hyperboloids. Outside P the two conicoids whenever real are both ellipsoids, or hyperboloids of one or two sheets.*" The case when two or more of the ratios were equal or λ_0 equal to one of these could be similarly treated.

7. If the two conicoids to start with are both paraboloids we may take both like elliptic paraboloids. If $a/a' > b/b' > c/c' > d/d'$ the three parabolids through a point are always real and two of them elliptic, the third hyperbolic. If however $a/a' > b/b' > d/d' > c/c'$ one of them is always real and is a hyperbolic paraboloid. The other two, whenever real, are both elliptic or both hyperbolic.

METHODS OF CONSTRUCTING MAGIC SQUARES

BY

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I

A method for constructing Magic Squares of the Concentric type having an odd number of cells in each row.

1. We gave a method for constructing Magic Squares of the concentric type, having an even number of cells in each row, in Vol. IV of this journal. We give below a method applicable to similar magic squares, having an odd number of cells in each row. The symbols K_n and S_n and terms such as " n - cell-square," complement of a number" etc., have the same meaning here as in the earlier paper.

2. The method we give below consists of first filling in numbers in the outermost belt, then in the one next to it and so on until we come to a 3 - cell square, "the Basic Magic Square." The same method, with slight modifications is used to fill up every one of the belts. But this method will not be applicable to the Basic Magic Square, which, however, may be constructed by the usual method of constructing a magic square of an odd order.

3. To Fill up the outermost belt of a $(2n+1)$-cell-square.

The outermost belt of a $(2n+1)$ - cell-square consists of $(2n+1)^2-(2n-1)^2$ or $8n$ cells. These cells have to be filled up by the first $4n$ natural numbers and their complements (got by subtracting from K_{2n+1}). This leaves the $(2n-1)^2$ numbers commencing from $4n+1$ and ending with $4n^2+1$ to be filled in the $(2n-1)$ - cell-square which is inside the belt.

Now let us write the numbers 1 to $4n$ in two rows so that there are an equal number of them in each row, as given below :—

$$\begin{array}{llcl}
A & \underline{1,\ 2,\ 3,\ 4,\ \cdots\cdots\cdots,\ n,} & \Big| & \underline{n+1,\ n+2,\ \cdots\cdots\cdots,\ 2n,} \quad B \\
C & 2n+1,\ 2n+2,\ \cdots\cdot,\ 3n, & \Big| & 3n+1,\ \cdots\cdots\cdot,\ 4n-1,\ 4n. \quad D
\end{array}$$

* The author is very much indebted to his friend Vidwan K. Lakshminarayana Srowthy of Sri Chamarajendra Vedamahapatasala, Bangalore for much help in the preparation of the paper and the testing of the method.

They are to be divided into four equal groups A, B, C and D as shown above. The first number in group C i.e. $2n+1$ is placed in the topmost cell of the right column of the belt and the second number from the end in group D i.e. $4n-1$ is placed in the bottommost cell of the same column. All the remaining numbers in group C excepting the last, along with the second number in group B i.e. $n+2$, are placed anywhere in this column. The numbers in group A are put in the left column of the belt so that they become the beginning numbers of rows in which there is no number in the end.

Now we come to the top and bottom rows. All the numbers in group B, excepting the second number which has already been placed elsewhere, may be placed in any of the cells of the top row, excepting the first cell. The first cell of the bottom row is not to be filled up at present and the last cell is, already, filled up. The last number in group C i.e. $3n$ and all the numbers of group D, excepting the one which is already placed elsewhere, are placed in the n cells of the bottom row, which are exactly below the unoccupied cells of the top row.

Now, we have filled up half of the cells in this belt. The other half is to be filled up with the complements of the numbers already put in. The complements † of $2n+1$ and $4n-1$ are put in the other end of the diagonals in which they are situated. The complements of other numbers are put in the other end of the row or column in which they are placed. Now the outermost belt is filled up.

The sum of the numbers first placed (i.e. excluding the complements of the numbers, introduced later on) in the right column is

$$(2n+1)+(2n+2)+\cdots+(3n-1)+(4n-1)+(n+2)=\frac{5n(n-1)}{2}+5n+1$$
$$=(5n^2+5n+2)/2.$$

The sum of the numbers first placed in the left column is

$$1+2+\cdots+n=(n^2+n)/2.$$

The sum of the numbers first put in the top row is

$$(n+1)+(n+3)+(n+4)+\cdots+2n+(2n+1)=n+1+\frac{(3n+4)(n-1)}{2}$$
$$=(3n^2+3n-2)/2; \text{ and}$$

the sum of the numbers first placed in the bottom row is

$$3n+(3n+1)+\cdots+(4n-2)+(4n-1)+4n=\frac{7n(n+1)}{2}=\frac{7n^2+7n}{2}.$$

† As we are dealing with numbers in a $(2n+1)$-cell square the complement of any number is got by subtracting it from K_{2n+1} i.e. $(2n+1)^2+1$.

Hence the sum of all the numbers in the right column is equal to $\frac{1}{2}(5n^2+5n+2)$ + sum of the Complements of the n numbers in the left column,

$$= \frac{5n^2+5n+2}{2} + \left[nK_{2n+1} - \frac{n^2+n}{2}\right] = nK_{2n+1} + (2n^2+2n+1)$$

$$= \left(n+\frac{1}{2}\right)K_{2n+1} = S_{2n+1}.$$

Similarly the sum of the numbers in the left column is

$$\frac{n^2+n}{2} + \left[(n+1)K_{2n+1} - \frac{5n^2+5n+2}{2}\right] = (n+1)K_{2n+1} - (2n^2+2n+1)$$

$$= \left(n+\frac{1}{2}\right)K_{2n+1} = S_{2n+1}.$$

The sum of the numbers in the top row is

$$\frac{3n^2+3n-2}{2} + \left[(n+1)K_{2n+1} - \frac{7n^2+7n}{2}\right] = (n+1)\,K_{2n+1} - (2n^2+2n+1)$$

$$= \left(n+\frac{1}{2}\right)K_{2n+1} = S_{2n+1};$$

and the sum of the numbers in the bottom row is

$$\frac{7n^2+7n}{2} + \left[nK_{2n+1} - \frac{3n^2+3n-2}{2}\right] = nK_{2n+1} + (2n^2+2n+1) = \left(n+\frac{1}{2}\right)K_{2n+1}$$

$$= S_{2n+1}.$$

Thus the sum of the numbers in each of the two rows and columns of the belt is S_{2n+1} as it should be. In each of the columns except the right and left columns and in each of the rows other than the top and bottom rows, as well as in each of the diagonals we have placed only two numbers whose sum is K_{2n+1}.

4. To fill up the cells in the remaining belts.

The above method is used to fill up the remaining belts also, until we come to the Basic Magic square. But as we have already used the first $4n$ numbers and their complements in the first ‡ belt we begin with $4n+1$ instead of with 1 in filling up the second belt. This belt will be the same as the outermost belt of a $(2n-1)$-cell-square, every number in which is increased by $4n$. When finding the complement of a number, the number should always be subtracted from K_{2n+1}, irrespective of the belt in which the number is placed.

After thus filling up all the belts, we fill up the basic magic square with the numbers $2n^2+2n-3$, $2n^2+2n-2$, $\cdots$, $2n^2+2n+5$.

5. We can easily prove that the diagonals as well as the rows and columns, not taken into consideration in § 4 above, also give the total S_{2n+1}.

‡ In the earlier paper the belts were numbered from inside, but here they are numbered from the outside.

If we remove the outermost belt of a $(2n+1)$-cell-square constructed by the above method, we get a $(2n-1)$-cell-square and the linear total of this square will e $\frac{(2n-1)\,S_{2n+1}}{(2n+1)}$. In general if we remove the $(n-\gamma)$ outermost belts of a $(2n+1)$-cell-square, the resulting $(2\gamma+1)$-cell magic square will have $\frac{(2\gamma+1)\,S_{2n+1}}{(2n+1)}$ for its linear total and so, if we subtract $\left(\frac{2\gamma+1}{2n+1}\,S_{2n+1}-S_{2\gamma+1}\right)\div(2\gamma+1)$ i.e. $2[(n^2+n)-(\gamma^2+\gamma)]$ from each of the numbers of this new magic square it will become a $(2\gamma+1)$-cell magic square containing the natural numbers 1 to $(4\gamma^2+4\gamma+1)$.

6. A 11-cell-square, constructed by the above method is given below as an illustration.

The Outermost or First belt

A 1, 2, 3, 4, 5, | 6, 7, 8, 9, 10, B
C 11, 12, 13, 14, 15, | 16, 17, 18, 19, 20. D

11 (the first no. in group C) and 19 (the second number from the end in D) are placed in the topmost and bottom-most cells respectively of right column. 12, 13, 14 (the remaining nos. of C, excepting the last) and 7 (the second number in B) are placed in right column.

1, 2, 3, 4, 5 (the nos. in A) are placed in left column. 6, 8, 9, 10 (the remaining numbers in B) are placed in the top row.

15 (last no. in C) and 16, 17, 18, 20 (the remaining numbers of D) are placed in the bottom row.

The complements are got by subtraction from $K_{11}=11^2+1=122$.

The Second belt

A 21, 22, 23, 24, | 25, 26, 27, 28, B
C 29, 30, 31, 32, | 33, 34, 35, 36. D

29 (first no. in C) and 35 (the second no. from the end in D) are placed in the top and bottom cells of right column. 30, 31 (the remaining nos. of C, excepting the last no.) and 26 (the second no. in B) are put in the right column. 21, 22, 23, 24 (nos. in A) are placed in the left column. 25, 27, 28 (the remaining nos. in B are placed in the top row. 32 (last no. in C), 33, 34, 36 (the remaining nos. in D) are placed in the bottom row.

The Third belt

A 37, 38, 39. | 40, 41, 42, B
C 43, 44, 45, | 46, 47, 48. D

43 (first no. in C) and 47 (second no. from end in D) are placed in the top and bottom cells of right column. 44 (the remaining no.

in C after excluding the last no.) and 41 (second no. in B) are put in the right column. 37, 38, 39 (nos. in A) are placed in left column. 40, 42 (the remaining nos. in B) are placed in top row. 45 (last no. in C), 46, 48 (the remaining nos. in D) are placed in the bottom row.

The Fourth belt

A 49, 50, | 51, 52. B } 53 (first no. in C) and 55 (second no.
C 53, 54, | 55, 56. D } from end in D) are placed in top right and bottom right corner cells. As no number but the last one remains in C, 52 (second no. in B) is put into a cell in the right column. 49 and 50 (nos. in A) are placed in left column. 51 (the remaining no. in B) is placed in top row. 54 (last no. in C) and 56 (the remaining no. in D) are placed in the bottom row.

The other cells in the belts are filled up with the complements of the above numbers and the § basic magic square with the numbers 57-65. The resulting magic square is reproduced below :—

103	6	8	9	10	107	106	105	104	102	11
110	87	90	89	88	86	25	27	28	29	12
109	92	75	77	76	74	40	42	43	30	13
108	91	78	67	68	66	51	53	44	31	14
115	96	81	70	64	57	62	52	41	26	7
5	21	37	49	59	61	63	73	85	101	117
4	22	38	50	60	65	58	72	84	100	118
3	23	39	69	54	56	71	55	83	99	119
2	24	79	45	46	48	82	80	47	98	120
1	93	32	33	34	36	97	95	94	35	121
111	116	114	113	112	15	16	17	18	20	19

II

A method of constructing 4n-cell-magic squares

1. In his book ‖ "Mathematical recreations and Essays", Ball has given two methods of constructing magic squares of order $4n$.

§ The remaining 3-cell square is the basic magic square. See § 2 above.

‖ Fourth Edition 1905 (MacMillan & Co.) pages 149-152.

According to the first method, a subsidiary square is formed whose cells are filled up with the numbers 1 to 16 n^2, commencing with the first cell in the top row and proceeding from left to right, taking up the second row after the first and so on. The subsidiary square is magic in the diagonals. Ball has shown that this square can be made magic in rows and columns by interchanging $4n^2$ of the numbers in the subsidiary square with the numbers in the complementary ¶ cells. Thus $8n^2$ numbers change their places, while the rest remain where they were first placed. The $8n^2$ cells, the numbers in which are to be thus changed, must be so selected that exactly $2n$ of them are to be in every row or column.

Ball gives two methods of selecting these cells. Here we propose to give a ** third method.

2. Each of the diagonals of the square passes through 2 quadrants. The cells in these two diagonals, along with those cells which are diagonally related to them, being also situated in the quadrants through which the respective diagonals pass, form one group. Or if we imagine the cells in the square to be coloured black and white alternately as in a chess-board,†† the black cells in the first and third quadrants along with the white cells in the second and fourth quadrants form one group. The remaining cells form another group, i.e.,

1st, 3rd, 5th, . . . , $(2n-1)$th, $(2n+2)$th, $(2n+4)$th, etc . . . . $4n$th cells in the rows of the same no. (i.e. 1st, 3rd etc. rows) and 2nd, 4th, 6th, . . . , $2n$th, $(2n+1)$th, $(2n+3)$th, . . . , $(4n-1)$th cells in the remaining rows form one group and the rest another.

The numbers in the cells of one group are left unchanged but the numbers in the cells of the other group are replaced by their complements (got by subtracting from K_{4n}) or, in other words, by the numbers in the complementary cells. The resulting square is magic.

¶ A $4n$-cell square may be divided into 4 equal squares by drawing a thick line between the $2n$th and $(2n+1)$th columns as well as between the $2n$th and $(2n+1)$th rows. We may call these 4 squares by the name "Quadrants" and the points where the four quadrants meet may be called the origin (as in a graph). Any two cells which are similarly situated with reference to the dividing lines are called complementary cells; i.e., the pth cell in the qth row is complementary to the $(4n-p+1)$th cell in the $(4n-q+1)$th row. In the subsidiary square above, complementary cells contain numbers which are complements to each other.

** For a 4-cell square this method gives the same result as Ball's method.

†† In the squares black coloured cells may be denoted by small dots put in the cells. This makes selection of cells easy.

3. We may prove, as follows, that it is a magic square:

Any row contains, exactly, $2n$ pairs of horizontally-related ‡ cells. The sum of the numbers in any such pair of cells in the rth row is $4n(2r-1)+1$. n out of these $2n$ pairs belong to one of the groups (stated above) and the rest to the other. So the sum of the numbers in this row, which belong to each of the groups is same; and, hence, the sum of their complements also must be the same. So after the interchanges are over, i.e. in the magic square, each row will contain $2n$ "original" numbers and $2n$ other numbers, whose sum is equal to the sum of the complements of the $2n$ "original" nos. in the row. So the sum of all the nos in any row is $2n\,K_{1n}$ i.e. S_{1n} as it should be. Similarly, it can be proved that the sum of the numbers in any column is also S_{1n}.

4. A 8-cell magic square has been constructed by this method and given below as an illustration.

1st, 3rd, 6th and 8th cells in the first, third, sixth and eighth rows and the 2nd, 4th, 5th and 7th cells in the second, fourth, fifth and seventh rows form one group. The numbers in the cells of this group are unaltered but the numbers in the other cells have been replaced by their complements.

The numbers, which are unaltered, are printed in ordinary type and the numbers, which have replaced their complements in italics.

1	*63*	3	*61*	*60*	6	*58*	8
56	10	*54*	12	13	*51*	15	*49*
17	*47*	19	*45*	*44*	22	*42*	24
40	26	*38*	28	29	*35*	31	*33*
32	34	*30*	36	37	*27*	39	*25*
41	*23*	43	*21*	*20*	46	*18*	48
16	50	*14*	52	53	*11*	55	*9*
57	*7*	59	*5*	*4*	62	*2*	64

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‡ Two cells in the same row, which are at equal distances from the two ends i.e. the mth and $(4n-m+1)$th cells are said to be horizontally related.

NOTES AND DISCUSSIONS

Lecture Notes

I. *Eccentricity of a Hyperbola*

To clear the ambiguity in the values of the eccentricity of a hyperbola given by the general equation, we offer the following tip:

THEOREM: *If* $ax^2+2hxy+by^2+2gx+2fy+c=0$ *be a hyperbola referred to axes inclined at an angle* ω, *the eccentricity corresponding to the real foci is greater than, equal to, or less than* $\sqrt{2}$ *according as* $\Delta\ (a+b-2h\cos\omega)$ *is greater than, equal to, or less than* 0, *where* Δ *is the usual invariant.*

The proof is immediate, by invariants.

For, if the canonical form be $a'x^2+b'y^2+c'=0$, and e the eccentricity corresponding to the real focus, e^2 is given by

$$1-\frac{a'}{b'} \text{ or } 1-\frac{b'}{a'} \text{ according as } a'c' \text{ is negative or positive.}$$

Let $a'c'$ be negative; then $b'c'$ is positive.

$$\text{Then } e^2-2 \gtrless 0$$

$$\text{according as } \frac{a'+b'}{b'} \lessgtr 0$$

$$\text{i.e. } \frac{a'+b'}{a'b'c'} \gtrless 0$$

$$\text{i.e. } \frac{a+b-2h\cos\omega}{\Delta} \gtrless 0 \quad \text{(by invariants)}$$

$$\text{i.e. } \Delta\ (a+b-2h\cos\omega) \gtrless 0$$

The same result is reached also when $a'c'$ is positive.

Cor. 1. An oblique hyperbola (i.e. not rectangular) is situated in the obtuse or acute angles between the asymptotes according as $\Delta\ (a+b-2h\cos\omega)$ is positive or negative.

Cor. 2. If Δ, Δ_1 denote the usual invariants corresponding to a hyperbola and its conjugate whose equations differ only in the constant term, $\Delta+\Delta_1=0$.

II. *The Jacobian of three conics touching one another at the same point*

The usual description of the above Jacobian is that it consists of a line and a conic (*Vide* p. 438, Askwith's *Analytical Geometry*, 1918). But the Jacobian actually consists of three concurrent straight lines made up of the common tangent at the point of contact and the double lines of the involution determined by the three pairs of common chords of the conics.

Taking Cartesian rectangular axes, with $y=0$ and $x=0$ for the tangent and normal at the point of contact, we write the equations of the three conics

$$S_r + 2 f_r y \equiv a_r x^2 + 2 h_r x y + b_r y^2 + 2 f_r y = 0 \qquad (r = 1, 2, 3)$$

and their pairs of common chords through the origin

$$C_i \equiv f_j S_k - f_k S_j = 0 \qquad (i \neq j \neq k, \; i, j, k = 1, 2, 3)$$

are evidently in involution, since $f_1 c_1 + f_2 c_2 + f_3 c_3 = 0$.

It is easily verified that the Jacobian of two pairs of lines through the origin is the same as the double lines of the involution determined by them. Hence, the double lines of the involution determined by the pairs of common chords are given by the Jacobian

$$\frac{\partial(C_i, C_j)}{\partial(x, y)} = 0$$

$$\text{i.e.} \quad \sum f_1 \cdot \frac{\partial(S_2, S_3)}{\partial(x, y)} = 0$$

But the Jacobian of three conics in question is

$$\begin{vmatrix} \dfrac{\partial S_1}{\partial x} & \dfrac{\partial S_1}{\partial y} + 2f_1 & 2f_1 y \\ \dfrac{\partial S_2}{\partial x} & \dfrac{\partial S_2}{\partial y} + 2f_2 & 2f_2 y \\ \dfrac{\partial S_3}{\partial x} & \dfrac{\partial S_3}{\partial y} + 2f_3 & 2f_3 y \end{vmatrix} = 0$$

$$\text{i.e.} \quad y \sum f_1 \frac{\partial(S_2, S_3)}{\partial(x, y)} = 0.$$

which proves our result.

III. *Lightning Elimination*

Problem: Eliminate (m, n) between

$$\frac{am^2+bm+c}{a'm^2+b'm+c'} = \frac{an^2+bn+c}{a'n^2+b'n+c'},$$

and $a''m^2+b''m+c'' = a''n^2+b''n+c'' = 0$

Note that (m, n) are the roots of two equations:

$$a''x^2+b''x+c''=0$$

$$(a-\lambda a')\,x^2+(b-\lambda b')\,x+c-\lambda c'=0$$

Hence $\dfrac{a-\lambda a'}{a''} = \dfrac{b-\lambda b'}{b''} = \dfrac{c-\lambda c'}{c''} = \mu$ (say)

Eliminating λ, μ, we get $\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} = 0.$

Application: To find the bisectors of the angles between the line-pair $ax^2+2hxy+by^2=0$, the angle between the axes being 'ω'.

The bisectors are given by

$$\frac{(y-m_1x)^2}{1+m_1^2+2m_1\cos\omega} = \frac{(y-m_2x)^2}{1+m_2^2+2m_2\cos\omega}$$

where $\quad ax^2+2hxy+by^2 \equiv b\,(y-m_1x)\,(y-m_2x)$

so that $\quad a+2hm_1+bm_1^2=0=a+2hm_2+bm_2^2$

Eliminating m_1, m_2 as in the original problem

$$\begin{vmatrix} x^2 & -xy & y^2 \\ 1 & \cos\omega & 1 \\ b & h & a \end{vmatrix} = 0$$

i.e. $\quad x^2(a\cos\omega-h)+xy\,(a-b)+y^2\,(h-b\cos\omega)=0.$

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Some Properties of Recurring Decimals

1. THEOREM: *If p, and q be primes, and the scale of notation R be prime to p and q, and if*

$$\frac{h}{p} = \cdot \dot{a}_1 a_2 \cdot \cdot \cdot \cdot \cdot \dot{a}_r$$

$$\frac{k}{q} = \cdot \dot{b}_1 b_2 \cdot \cdot \cdot \cdot \cdot \dot{b}_{rt}$$

then the period-number of $\frac{k}{q}$ that is the number represented in scale R by $b_1 b_2 \cdots b_{rt}$ is divisible by p.

Proof:— $a_1 a_2 \cdots a_r = (R^r - 1)h/p$

$b_1 b_2 \cdots b_{rt} = (R^{rt} - 1)k/q$

Now, $R^{rt} - 1$ is a multiple of $R^r - 1$ which is a multiple of p. Also $(p, q) = 1$. Hence $R^{rt} - 1$ is a multiple of p.

$\therefore$ The period-number of k/q is divisible by p.

Also, if $t = 1$, then applying the property to each divisor, we get that

the period-number of the first division is divisible by the second divisor and *vice versa*.

2. *If n is the highest power of p which divides $10^l - 1$ where $(l, p) = 1$ and r is a positive integer then* † $10^c \equiv 1 \pmod{p^{n+r}}$ *if $c = lp^r$ and the period-number of $1/p^n$ and of $1/p^{n+r}$ leave the same remainder, when divided by p.*

Proof:—Let $(10^l - 1)/p^n = M$ where $(M, p) = 1$.

Then $(10^c - 1)/p^{n+r} =$ an integer N if $(N, p) = 1$.

Further,

$$N = (10^c - 1)/p^{n+r} = \{(Mp^n + 1)^t - 1\}/p^{n+r}$$

$$= \{Mp^{n+r} + kp^{n+r+1}\}/p^{n+r}$$

$$= M + kp$$

when $t = p^r$ and k is an integer.

Hence $N \equiv M \pmod{p}$.

So, if $1/p^n = \cdot \dot{a}_1 a_2 \cdot \cdot \cdot \cdot \cdot \cdot \dot{a}_l$

and $1/p^{n+r} = \cdot \dot{b}_1 b_2 , \cdot \cdot \cdot \cdot \dot{b}_c$

† *Vide* Hardy and Wright. *An introduction to the Theory of Numbers* pp. 65, 66.

then $a_1 a_2 \cdots a_t \equiv b_1 b_2 \cdots b_t \pmod{p}$

Also, the most usual value of n is 1.

In all such cases, the period-number of $1/p$ and the period-number of $1/p^{r+1}$ differ by a multiple of p.

That is, the period-number of $p^{-x} \equiv$ period-number of $p^{-1} \pmod{p}$ for any integer x. The same principle holds essentially for any radix, which is prime to p.

§ 3. In this paragraph I shall explain a simple method of obtaining the decimal expansions of reciprocals of powers of a prime p. Let us take $p = 13$ by way of illustration.

By Fermat's Theorem $10^{12} \equiv 1 \pmod{13}$. Hence the number of decimal places n in $1/13$, i.e., the least number n such that $10^n \equiv 1$, (mod 13) is a factor of 12. Here $n = 6$ and in fact $\frac{1}{13} = \cdot\dot{0}7692\dot{3}$.

The period number (P. N) of 1/13 is thus $r = 076923$. Since it is an integer the 0 at the beginning makes no difference, but it is convenient to retain it.

The number of places in the period of $1/13^2$ is 13×6. We shall now obtain the corresponding P. N.

We first divide r by 13 giving 005917 which are the first 6 digits in $1/13^2$ and leaving a remainder 2. To get the next 6 places add 2×076923 to 005917 thus getting 159763. The remainder at this stage is 4. Continuing thus we have

	Remainder
0 0 5 9 1 7	2
1 5 3 8 4 6	
1 5 9 7 6 3	4
1 5 3 8 4 6	
3 1 3 6 0 9	6
1 5 3 8 4 6	
4 6 7 4 5 5	8
1 5 3 8 4 6	
6 2 1 3 0 1	10
1 5 3 8 4 6	
7 7 5 1 4 7	12
1 5 3 8 4 6	

	Remainder
9 2 8 9 9 3	$14 \equiv 1 \pmod{13}$
1 5 3 8 4 6	
1, 0 8 2 8 3 9	3
1 5 3 8 4 6	
1, 2 3 6 6 8 5	5
1 5 3 8 4 6	
1, 3 9 0 5 3 1	7
1 5 3 8 4 6	
1, 5 4 4 3 7 7	9
1 5 3 8 4 6	
1, 6 9 8 2 2 3	11
1 5 3 8 4 6	
1, 8 5 2 0 6 9	$26 \equiv 0 \pmod{13}$
1 5 3 8 4 6	
2, 0 0 5 9 1 5	

The period-number of $1/13^2$ is obtained by writing down the results of the successive additions. Where the sums have the additional digit 1 or 2 (separated by a comma) these are added to the last digit of the previous sum. Thus the period of $1/13^2$ is

$$\begin{array}{cccccc}
\dot{0} & 0 & 5 & 9 & 1 & 7 \\
1 & 5 & 9 & 7 & 6 & 3 \\
3 & 1 & 3 & 6 & 0 & 9 \\
4 & 6 & 7 & 4 & 5 & 5 \\
6 & 2 & 1 & 3 & 0 & 1 \\
7 & 7 & 5 & 1 & 4 & 7 \\
9 & 2 & 8 & 9 & 9 & 4 \\
0 & 8 & 2 & 8 & 4 & 0 \\
2 & 3 & 6 & 6 & 8 & 6 \\
3 & 9 & 0 & 5 & 3 & 2 \\
5 & 4 & 4 & 3 & 7 & 8 \\
6 & 9 & 8 & 2 & 2 & 4 \\
8 & 5 & 2 & 0 & 7 & \dot{1}
\end{array}$$

For $1/13^3$ the number of digits will be $6 \times 13^2 = 1014$ which we obtain, as before, in 13 sets of 78 digits. The first set is obtained by dividing the P. N. of $1/13^2$ say r' by 13. The remainder will be, according to § 3 the same as before, namely 2. So the next set of 78

digits is obtained by adding $2 \times 076923 = 153846$ to every successive group of 6 digits in the first set. The first 2 sets are given below :

First set		*Common difference*		*Second set*
0 0 0 4 5 5		1 5 3 8 4 6		1 5 4 3 0 1
1 6 6 1 3 5		1 5 3 8 4 6		3 1 9 9 8 1
6 3 9 5 0 8		1 5 3 8 4 6		7 9 3 3 5 4
4 2 0 5 7 3		1 5 3 8 4 6		5 7 4 4 1 9
5 0 9 3 3 0		1 5 3 8 4 6		6 6 3 1 7 6
9 0 5 7 8 0	+	1 5 3 8 4 6	=	1, 0 5 9 6 2 6
6 0 9 9 2 2		1 5 3 8 4 6		7 6 3 7 6 8
6 2 1 7 5 6		1 5 3 8 4 6		7 7 5 6 0 2
9 4 1 2 8 3		1 5 3 8 4 6		1, 0 9 5 1 2 9
5 6 8 5 0 2		1 5 3 8 4 6		7 2 2 3 4 8
5 0 3 4 1 3		1 5 3 8 4 6		6 5 7 2 5 9
7 4 6 0 1 7		1 5 3 8 4 6		8 9 9 8 6 3
2 9 6 3 1 3 Rem. 2		1 5 3 8 4 6		4 5 0 1 5 9 Rem. 4

The 1 preceding the comma should be added to the last digit of the previous row.

The 3rd and other sets may be obtained similarly. Thus, given P. N. of $1/d^n$ we can find P. N. of $1/d^{n+1}$ by adding successively to it a common difference obtained as follows:—

Multiply P. N. of $1/d$ by the remainder when it is divided by d and repeat this d^{n-1} times as in the above example.

Mayavaram.

M. S. SRINIVASAN.

REVIEWS

M. RAZIUDDIN SIDDIQI, M.A. (Cantab), Ph.D. (Leipzig). *Lectures on Quantum Mechanics.* Osmania University Publications, 1938 pp. 293.

This book gives an introductory account of old and new quantum mechanics and their elementary applications. After a brief survey of the essentials of the special theory of Relativity and the general dynamical theory with its Lagrangian and Hamiltonian functions, angle and action variables, canonical equations and contact transformations, the book proceeds to describe the Rutherford model of the atom followed by an account of the Quantum theory of radiation illustrating its particle nature by a consideration of the photoelectric and the Compton effects. Then Bohr's theory of the atom, the correspondence principle and Sommerfeld's relativity theory of the fine structure of spectral lines come up for detailed treatment.

A criticism of Bohr's theory and Heisenberg's quantum mechanics with a detailed account of matrix calculation, Poisson bracket, and angular momentum, illustrated by application to the harmonic oscillator, are dealt with in the succeeding chapter. This is followed by a description of de Broglie's wave idea of matter and Schrodinger's derivation of the differential equation for matter waves. The polynomial method of determining the *eigen* values with applications to the harmonic oscillator and the hydrogen-like atoms accompanied by consideration of the equivalence of the methods of Heisenberg and Schrodinger and the derivation of the matrix elements from *eigen* functions is clearly treated in the next chapter. Perturbation theory of the first and second orders for non-degenerate and degenerate cases with the Stark effect and the Helium atom as examples and the same theory in Heisenberg's mechanics are then considered in detail. The physical interpretation of the wave functions by Schrodinger and Born, Heisenberg's principle of indeterminancy relativistic quantum mechanics and Dirac's theory of the electron form the subject-matter of the closing chapters of the book.

Almost all the above topics have been dealt with before in numerous books on quantum mechanics. The particular value of this book lies in its abundance of expository detail. Many students of Physics are quite unfamiliar with the mathematical technique required for a proper understanding of the principles and applications of quantum mechanics. The required mathematical apparatus is almost completely given in the relevant portions of the book. Perhaps a brief account of Legendre and Laguerre polynomials and the different systems of orthogonal co-ordinates used in quantum mechanics may not be out of place in this book.

There are unfortunately quite a number of misprints which a more careful proof correction could have avoided. References to previous articles and equations are also not given correctly in several places.

The book is intended for readers with no extensive training in higher mathematics and is written with a good appreciation of the requirements of such readers. Clearness in exposition is attained by full explanations of the various steps in almost all the difficult mathematical calculations. The book is an excellent guide to those Physicists and Chemists, who desire to have a working knowledge of quantum mechanics. The students who read this book will be eagerly awaiting the publication of the second volume promised by the author in his preface.

S. NARAYANASWAMI IYER.

G. S. MAHAJANI: *The application of moving axes methods to the Geometry of Curves and Surfaces*, 1937, Aryabhushan Press, Poona, 60 pp.

Students of Mathematics (in which class I include firstly the teachers of the subject !), for whom the language barrier has stood in the way of an appreciation of the beautiful methods associated with the *trièdre mobile*, will be grateful to Dr. Mahajani for this elementary introduction to the subject.

After a brief explanation of the vectorial notation, the author obtains in the first chapter the fundamental formulae connecting the rates of change of a vector referred to fixed and to moving frames of reference.

In the second chapter of 26 pages we have a fairly complete account of the better known differential properties of space curves *viz.*, Frenet's Formulae, the developables associated with a curve, spherical curvature, Bertrand Curves etc. While most English text-books give a left-handed treatment of the subject, the author uses right-handed reference frames (equivalent to a reversal of the sense of the binormal) and obtains as the reward of his chivalry a more symmetric distribution of signs in Frenet's Formulae.

The third chapter deals briefly with the frames associated with a surface and includes Meunier's and Euler's Theorems, geodesic curvature and torsion, lines of curvature, Asymptotic lines and Joachimsthal's Theorem. Chapter IV which is devoted to applications deals with curves on the sphere, the cone and the cylinder. There is a collection of 10 examples given at the end while there are a few given in the body of the book, mostly taken from Tripos papers.

The book is apparently not intended to be a text-book on Differential Geometry; for how can one deal adequately with the geometry of "Curves on surfaces" in a ten-page chapter? However, to the student who is already acquainted with the main results of the subject by the usual methods, Dr. Mahajani's book offers the stimulating experience of traversing the same field in a new vehicle and of recognising the same landmarks from a new perspective. The author's clarity of exposition and helpful remarks ensure that the reader's vision of the domain from his moving axes shall be bright and clear.

A. NARASINGA RAO.

SOLUTIONS TO QUESTIONS

Question 1654

(S. CHOWLA):—Given any $\alpha > 0$, is it true that the numbers $\log(1+p^{-\alpha})$ where p runs through a finite number of primes are linearly independent *i.e.* there is no relation of the form

$$\sum c_p \log(1+p^{-\alpha}) = 0$$

where the numbers c_p are integers?

Prove this result in the special case when α is a positive rational number < 1.

Solution by Proposer

Let $\alpha = c/d$ where $(c, d) = 1$. Let

$$\prod_p (1+p^{-c/d})^{c_p} = 1 \quad \text{or} \quad \prod_p (p^{c/d}+1)^{c_p} = \prod_p (p^{c_p \cdot c/d}). \qquad \ldots \quad (1)$$

Raising both sides to the power d, we easily see that (1) is impossible—the left side is irrational, the right is rational. The case $\alpha = 1$ needs separate treatment.

Question 1686

(A. A. KRISHNASWAMI AYYANGAR):—If $\int_a^b x^n f(x) = 0$ for integral $n > 0$, is $f(x)$ identically zero?

Solution by the proposer

If (a, b) is a finite interval, $f(x)$ is zero except for a set of measure zero and is identically zero if $f(x)$ is continuous. If (a, b) is an infinite interval, the result is no longer true even if $f(x)$ be continuous. Examples, in this connection, are :—

(i) $\int_0^\infty x^n[\exp(x^\mu \cos \mu\pi)] \sin(x^\mu \sin \mu\pi) = 0$ for $n = 0, 1, 2, \cdots$ and $0 < \mu < 1/2$.

(ii) $$\int_0^\infty x^n \left[\exp\left(-\frac{\pi\sqrt{x} - \log x}{(\log x)^2+\pi^2}\right)\right] \sin \frac{\sqrt{\pi}\,\log x + \pi}{\pi^2+(\log x)^2}\, dx = 0.$$

(For these examples see Polya and Szego, *Aufgaben and Lehrsatze* Abschitt 2, Questions 138, 139, 142. A simpler example is to be found in Titahmarsh, *Theory of Functions*, p. 106-107.)

(iii) $\int_0^\infty x^n \exp(-x^{\frac{1}{4}}) \sin x^{\frac{1}{4}} = 0, \qquad n = 0, 1, 2, 3, \cdots$

Question 1707

(A. A. KRISHNASWAMI AYYANGAR):—A, A′ denote two integers such that the order of the digits in one is reversed in the other. Investigate the existence of numbers which can be represented in more than one way as the product of two such members A and A′. Eg. $3472 \times 2743 = 1456 \times 6541$.

Solution by M. Sitaramanjanujulu

I give below a general rule for obtaining pairs of nos. such that $AA' = BB'$.

Let x and y be two nos. each with any no. of digits. Let x' and y' be the nos. formed by reversing the digits in x and y respectively. The nos. x and y should be such that they satisfy the following conditions.

(i) Any digit in x when multiplied by any digit in y should not give a number greater than 9.

(ii) When x is multiplied by y or y' as in the ordinary multiplication in Arithmetic (digit by digit as shown below), the sum of the digits in any column should not exceed 9. (This includes indirectly case (i) also).

Then $\quad (x \cdot y) \times (x' y') = (x \cdot y') \times (x' \cdot y)$

xy and $x'y'$ give a pair of nos. like $A + A'$ and $x \cdot y'$ and $x' \cdot y$ give another pair.

For example

$$(21121 \times 1112)(12112 \times 2111) = (21121 \times 2111)(12112 \times 1112)$$

or $\quad 23486552 \times 25568432 = 44586431 \times 13468544$

Actual multiplication as given below makes the above reasoning clear.

(x) 21121 (y) 1112	(x') 12112 (y') 2111	(x) 21121 (y') 2111	(x') 12112 (y) 1112
42242 21121 21121 21121	24224 12112 12112 12112	42242 21121 21121 21121	24224 12112 12112 12112
23486552	25568432	44586431	13468544

N. B.—The case when $x = x'$ or $y = y'$ is trivial.

e.g., $\quad 132 \times 231 = 131 \times 231.$

Question 1715

(A. A. KRISHNASWAMY AYYANGAR):—When two integers are multiplied together, show that the digit in the highest place in the product can never be intermediate between the corresponding digits in the given factors.

Solution by A. Ranganatha Rao

Let F_1, F_2 be the factors and P the product, f_1, f_2, p the digits in their highest places and n_1, n_2, m the number of digits in F_1, F_2, P, respectively. Then it is easy to see that $m = n_1 + n_2$ or $n_1 + n_2 - 1$. We exclude the case when $p = f_1$ or f_2, for then, nothing remains to be proved. We visualise the process of division to be actually performed with P as the dividend and F_1, say, as the divisor, so that F_2 is the quotient. Considering the very first and the subsequent stages of the division, it is clear that the case $m = n_1 + n_2$ can arise only if $p < f_1$. Then, similarly regarding F_2 instead of F_1 as the divisor, p must be less than f_2 also. Again, it is shown in a similar way that the case $m = n_1 + n_2 - 1$ can arise only if $p > f_1$ and f_2. Hence the result.

Question 1739

(A. A. KRISHNASWAMY AYYANGAR):—Show that

$\int_0^x e^{-x^2/2} dx = \sqrt{\frac{\pi}{2}} \left(1 - e^{-mx^2}\right)$ when $m > \frac{1}{2}$ has only one real root in the interval $(0, \infty)$. If α_m is the root, prove that

(i) $\alpha_m \to 0$ as $m \to \infty$

(ii) $\int_0^x e^{-x^2/2} dx > \sqrt{\frac{\pi}{2}} \left(1 - e^{-mx^2}\right)$ in $0 < x < \alpha_m$

and (iii) $\int_0^x e^{-x^2/2} dx < \sqrt{\frac{\pi}{2}} \left(1 - e^{-mx^2}\right)$ when $x > \alpha_m$.

Additional Solution by N. Vencoba Rao

Let $f(x) = \int_0^x e^{-x^2/2} dx - \sqrt{\frac{\pi}{2}} \left(1 - e^{-mx^2}\right)$

Then $f'(x) = e^{-mx^2} \left[e^{(m-1/2)x^2} - \sqrt{2\pi} \cdot mx \right]$

Let $\phi(x) = e^{(m-1/2)x^2} - \sqrt{2\pi} \cdot mx$; so that

$$\phi'(x) = e^{(m-1/2)x^2}\, 2(m-1/2)x - m\sqrt{2\pi}$$

$$\therefore \quad \phi''(x) = 2 \cdot (m-1/2)\, e^{(m-1/2)x^2} \left\{ 1 + 2\,(m-1/2)x^2 \right\}$$

which for $m > 1/2$ is always positive.

So $\phi'(x)$ is always increasing and hence has only one root β. (i.e.) $\phi'(x)$ is negative in $(-\infty, \beta)$ and positive in (β, ∞).

It can be verified that $\phi'(1/\sqrt{2m-1})$ is negative and so $\beta > 1/(\sqrt{2m-1})$; but β is given by $e^{(m-1/2)\beta^2} \cdot 2\,(m-1/2)\,\beta = m\sqrt{2\pi}$

$\therefore\ \phi(\beta) = e^{(m-1/2)\beta^2} - e^{(m-1/2)\beta^2} \cdot (2m-1)\beta^2$ which when $\beta^2 > 1/(2m-1)$ and $m > 1/2$ is negative and $\phi(0)$ is positive.

So $\phi(x)$ and hence $f'(x)$ is decreasing in $(0, \beta)$ and increasing in (β, ∞); also $f'(\beta)$ is negative. So $f'(x)$ has two and only two real roots say (λ, μ).

It follows that $f(x)$ is increasing in $(0, \lambda)$, decreasing in (λ, μ) and increasing in (μ, ∞). The graph of $f(x)$ in $(0, \infty)$, rises from the origin, to a certain maximum value at $x = \lambda$, descends in (λ, μ) and again rises asymptotically towards the x-axis. Since $f(\infty) = 0$, and $f(x)$ is increasing in (μ, ∞) $f(\mu)$ is negative. So $f(x)$ vanishes once in (λ, μ) say at a_m. The results (ii) and (iii) of question 1739 now follow.

(i) λ and μ are given by $e^{(m-1/2)x^2} = mx\sqrt{2\pi}$.

Taking logarithms of both sides it follows that

$$x^2 = \frac{\log\sqrt{2\pi}}{m-\frac{1}{2}} + \frac{\log m}{m-\frac{1}{2}} + \frac{\log x}{m-\frac{1}{2}}$$

and hence $\lim\limits_{m\to\infty} x^2 = 0$; i.e. as $m \to \infty$, λ and $\mu \to 0$ and so $a_m \to 0$.

Question 1755

(B. R. VENKATARAMAN):—The necessary and sufficient condition that the F and ϕ conics of two conics S, S′ be concentric is that S and S′ be concentric. In this case the four conics have the same centre and the asymptotes of the four conics belong to the same involution pencil.

Solution by the Proposer and A. Ranganatha Rao.

It is known that all the four conics in question have a common self-polar triangle. If any two of the conics are concentric, the common centre and the line at infinity constitute a vertex and the

opposite side of the common self-polar triangle of these two conics, and therefore of all the four conics. Again, the remaining two sides of the self-polar triangle are conjugate diameters for each of the conics; hence these lines separate harmonically the asymptotes of the four conics which must, therefore, belong to an involution pencil.

We may also note that since two parabolas with parallel axes are concentric conics their F and ϕ conics are also parabolas whose axes are parallel to those of the given parabolas.

QUESTIONS FOR SOLUTION

1780. (M. S. SRINIVASAN):—If p be prime and $M = 1\ 3\ 5 \cdot 7 \cdots (p-2)$ prove that $M^4 \equiv 1 \pmod{p}$.

1781. (P. KESAVA MENON):—Show that

$$\log (n+1) - e \sum^{n} \{r^r/(r+1)^{r+1}\}$$

tends to a definite limit which lies between zero and $2(1-\log 2) - \gamma$ when $n \to \infty$, γ being Euler's constant.

1782. (P. KESAVA MENON):—If P denotes the perimeter of the ellipse whose semi axes are a and b respectively, show that

$$\pi \sqrt{2a^2 + 2b^2} > P > \pi\,(a+b)$$

and hence deduce that when a and b are very nearly equal, the perimeter may be taken to be $\pi(a+b)$.

1783. (A. NARASINGA RAO):—From a projective space P_n of n dimensions, an open set of points P'_n is obtained by removing all points on a straight line in P_n. What are the topological properties of the set P'_n?

1784. (D. KRISHNA RAO):—Show that

$$\iiiiiint \sum \delta p_i\ \delta q_i\ \delta p_j\ \delta q_j\ \delta p_k\ \delta q_k$$

where the summation is extended over the $n(n-1)(n-2)/6$ combinations of the indices i, j, k is an integral invariant of any Hamiltonian system in which q_i, p_i are the variables.

ELEVENTH CONFERENCE OF THE INDIAN MATHEMATICAL SOCIETY

OSMANIA UNIVERSITY—HYDERABAD (DECCAN)

December 1939.

Sitting (Left to Right)— 1. Prof. M. V. Arunachala Sastry. 2. Prof. Mohd. Abdur Rahman Khan. 3. Prof. A. Narasinga Rao. 4. Dr. R. Vaidyanathaswamy. 5. Mohd. Azhar Hasan, Esq., *Secretary, Home Dept. H. E. H. The Nizam's Govt.* 6. Dr. N. R. Sen. 7. Nawab Mehdi Yar Jung Bahadur, *Education Member and Vice-Chancellor, Osmania University.* 8. Prof. Qazi Mohammad Husain, *Pro-Vice-Chancellor, Osmania University.* 9. Prof. Ram Behari. 10. Prof. F. W. Levi. 11. Dr. T. Vijayaraghavan. 12. Prof. K. S. K. Iyengar. 13. Dr. B. R. Seth. 14. Prof. A. N. Singh.

Standing (1st Row)— 1. Prof. S. Mahadevan. 2. V. Narasimha Murti. 3. Prof. Waheedur Rahman. 4. Dr. M. R. Siddiqi. 5. Prof. Karamchand Dhawan. 6. Prof. Abdul Haq. 7. Prof. A. H. Quraishi. 8. Prof. A. L. Shaik. 9. Prof. K. G. Shah. 10. D. R. Kaprekar, Esq. 11. Prof. V. Rangachariar. 12. Prof. S. A. Hameed. 13. Dr. M. Ziauddin. 14. Prof. S. K. Abhayankar. 15. Prof. K. D. Panday. 16. Prof. M. K. Kevalramani.

Standing (2nd Row)— 1. Prof. Vaidyanatha Sastry. 2. Prof. C. R. Chaturvedi. 3. S. Meenakshisundaram, Esq. 4. Prof. K. Sambasiva Rao. 5. Dr. S. S. Pillai. 6. Prof. V. Seetharaman. 7. Dr. B. Ramamurti. 8. Fr. C. Racine. 9. Prof. K. Satyanarayana. 10. Prof. P. Shintre. 11. Prof. Godhole. 12. Prof. R. Krishnamurti. 13. Prof. L. N. Subramanyan. 14. Dr. B. S. Madhava Rao. 15. Prof. John Maclean. 16. Dr. N. G. Shabde.

Standing (3rd Row)— 1. Student. 2. Dr. W. F. Kibble. 3. Dr. Qazi Syed Moinuddin. 4. Prof. Mohd. Akram, 5. Father Rafael. 6. Prof. E. Banerji. 7. Prof. K. S. Ramamurti. 8. Prof. K. V. Iyengar. 9. Muneeruddin, Esq. 10. Dr. J. C. Kameshwar Rao. 11. Prof. S. V. Bhagwat. 12. Prof. K. R. Gunjikar. 13. Prof. Kishen Chand. 14. Prof. N. K. Trivedi. 15. Prof. B. B. Bagi. 16. Prof. Mohd. Ali.

Standing (4th Row)— 1 to 11. Students. 12. Prof. T. Venkatarayudu. 13. Student. 14. Student. 15. Mohd. Abdul Aziz, Esq. 16 Student. 17. Prof. V. P. Venkatachari. 18. Student. 19. Prof. Mohd. Khwaja Mohiuddin. 20. Prof. D. D. Kosambi. 21. Student. 22. Syed Wahajuddin Ahmad, Esq. 23—32. Students.

THE MATHEMATICS STUDENT

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The Eleventh Conference

OF THE

INDIAN MATHEMATICAL SOCIETY

The Indian Mathematical Society was invited by the Osmania University Hyderabad, Deccan, and held its Eleventh Conference on the 21st, 22nd and 23rd of December 1939 in the palatial buildings of the University Arts College.

His Exalted Highness the Nizam was pleased to send a Message of Good Wishes which was read out by the *Hon'ble Nawab Mehdi Yar Jung Bahadur*, Member for Education and Vice-Chancellor, Osmania University, who then declared the Conference open. *Prof. Qazi Mohammad Husain*, Pro-Vice-Chancellor, Osmania University, welcomed the delegates. The Inaugural Address was delivered by *Dr. N. R. Sen*, Ghosh Professor of Applied Mathematics, University of Calcutta.

There were two symposia, one on "Generalised Geometry including Relativity and Field Theories" presided over by *Prof. D. D. Kosombi* of Poona, and the other on "Waring's Problem" under the presidency of *Dr. S. S. Pillai* of Annamalainagar. There were three evening discourses, the first on "Meteoric Astronomy" by *Prof. M. Abdul Rahman Khan*, Hyderabad, the second on "Mathematical Puzzles and Recreations" by *Prof. A. Narasinga Rao* of Annamalainagar and the third on "Mathematics, the Handmaid of Arts, Science and Economics" by *Dr. T. Vijayaraghavan* of Dacca.

As usual there was a Business Meeting and a Discussion on the Teaching of Mathematics in Schools and Colleges. The place of Mathematics in competitive examination in India was one of the subjects which came up for comment.

Nearly 60 papers were contributed to the Conference. In the absence of Dr. R. P. Paranjpye, President of the Society, Prof. N. R. Sen presided at the meetings of the Conference.

The visitors and delegates were treated to an "At Home" on Thursday the 21st December and to a sumptuous dinner on Friday the 22nd December. The Toast of H. E H. The Nizam was proposed by Prof. Qazi Mohd. Husain. The toast of the guests was proposed by Prof. Kisen Chand and was responded by Prof. N. R. Sen.

The excursions included a visit to the local Observatory, the Golconda Fort and tombs, the Osman Sagar and a 2-day trip to Daulatabad and the rock-hewn caves of Ajanta and Ellora.

The delegates were comfortably lodged in the University Hostel and the catering and other arrangements made by the Reception Committee were highly appreciated by the visitors.

Message from His Exalted Highness the Nizam, to the Eleventh Conference of the Indian Mathematical Society

I am glad to have this opportunity of sending a message of good wishes on the occasion of the Eleventh Session of the Indian Mathematical Conference which is being held in the Capital of my Dominions. I hope you will find your stay here pleasant and that your deliberations will be conducive to the advancement of your great and noble science.

Mathematics is one of the most ancient and highly developed branches of knowledge, and it is a matter of pride to all of us that both the Hindus and the Muslims have contributed a great deal to its early development. I am sure that in the midst of your labours you will find time to see something of this historic City, and that you will realize that our life and culture are just as much a common heritage of the Hindu and Muslim peoples as the Science of which you are the devotees.

I wish all success to your deliberations and hope that your concerted efforts will produce results calculated to bring about a far-reaching advancement of mathematical knowledge.

21st December, 1939.

Reception Committee

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Prof. Qazi Mohammad Husain,
Pro-Vice-Chancellor, Osmania University.

Secretary

Dr. M. Raziuddin Siddiqi, Professor of Mathematics, Osmania University.

Joint-Secretary

M. Khaja Mohiuddin Esq.,
Lecturer in Mathematics, Osmania University.

Members of the Reception Committee

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V. N. Patwari, Esq.
Qadir Husain Khan, Esq.
Raghavendar Rao, Esq.
Sajjad Mirza, Esq.
R. N. Satyanarayan, Esq.
Shaik Barkat Ali, Esq.
Syed Mohammad Younus, Esq.
Qazi Mohammad Husain, Esq.
Waheedur Rahman, Esq.
V. P. Venkatachari, Esq.
Raziuddin Siddiqi, Esq.

Daily Programme

Thursday, 21st December, 1939

10-30 a.m. Reading of the Message from His Exalted Highness the Nizam of Hyderabad.

Welcome Address by the Chairman of the Reception Committee, Prof. Qazi Mohammad Husain, Pro-Vice-Chancellor, Osmania University.

Inauguration of the Conference by the Hon'ble Nawab Mehdi Yar Jung Bahadur, Member for Education and Vice-Chancellor, Osmania University.

Report of the Indian Mathematical Society by Dr. Ram Behari, Secretary of the Society.

Inaugural Address by Prof. N. R. Sen, Ghosh Professor of Applied Mathematics, University of Calcutta.

Group Photo: Delegates and Members.

2 p.m. Symposium on "Generalised Geometry including Relativity and Field Theory."

4-30 p.m. Reception Committee "At Home."

6-30 p.m. Popular Lecture on "Meteoric Astronomy" by Prof. M. Abdul Rahman Khan, Hyderabad.

Friday, 22nd December, 1939

9 a.m. Reading of papers.

2 p m. Symposium on "Waring's Problem"

3-30 p m. Sightseeing Tour round the City.

6-30 p.m. Popular Lecture on "Mathematical Recreations" by Prof. A. Narasinga Rao, Annamalai University.

Saturday, 23rd December, 1939

9 a.m. Reading of papers.

12 noon. Business Meeting of the Indian Mathematical Society.

2 p.m. Discussion on the "Teaching of Mathematics."

6 p.m. Popular Lecture on "Mathematics, the Handmaid of Arts, Science and Economics" by Dr. T. Vijayaraghavan of the Dacca University.

7-30 p.m. Departure of delegates to Aurangabad for Excursion to Ellora and Ajanta Caves.

Delegates

Andhra University—
V. Narasimhamurti, Esq.
K. Sambasiva Rao, Esq.

Annamalai University—
A. Narasinga Rao, Esq.
S. S. Pillai, Esq.
V. Seetharaman, Esq.

Bombay University—
D. D. Kosambi, Esq.
N. M. Shah, Esq.

Calcutta University—
F. W. Levi, Esq.

Dacca University—
N. M. Basu, Esq.
T. Vijayaraghavan, Esq.

Delhi University—
Ram Behari, Esq.
B R. Seth, Esq.

Gwalior State—
S. K. Abhyankar, Esq.

Hindu University, Benares—
V. V. Narlikar, Esq.
Brij Mohan, Esq.

Madras University—
R. Vaidyanathaswamy, Esq.

Muslim University, Aligarh—
M Ziauddin, Esq.

Mysore University—
K. S. K. Iyengar, Esq.
B. S. Mahadeva Rao, Esq.
K. V. Iyengar, Esq.

Punjab University—
P. Samuels Lall, Esq.

Travancore University—
R. Srinivasan, Esq.
H. Subramania Iyer, Esq.

Opening Speech

BY

THE HON'BLE NAWAB MEHDI YAR JUNG BAHADUR,
Education Member, H. E. H. The Nizam's Government,
Vice-Chancellor, Osmania University.

DELEGATES, LADIES, AND GENTLEMEN,

On behalf of the Osmania University and the Government of Hyderabad I have great pleasure in extending to you a hearty welcome to this beautiful and historic city, the capital of the Premier State of India, where we are gathered together for the Eleventh session of the Indian Mathematical Conference. I hope your stay will be pleasant and comfortable in every way.

In point of years, our University is one of the youngest in India, but during the brief period of its existence, which does not exceed two decades, it has set itself the difficult task of demonstrating to the whole of India that it is not only possible and practicable but actually better to impart instruction in

the higher branches of knowledge (including Mathematics) through one of the principal Indian languages, in this case the *lingua franca* of India. The fact that the Osmania University degrees are now recognised by the British Universities of Oxford, Cambridge and London as well as by other Universities, and that our Medical College Diplomas are recognized by the Royal College of Surgeons, proves the success achieved by this medium of instruction. Our success has encouraged educationists in other Provinces to adopt the aim of imparting education through one or other of the principal languages of India.

This State is not without some title to historic fame. It has had a great past, and the world-renowned caves of Ellora and the frescoes of Ajanta, the forts of Daulatabad and the tombs of Golconda and a host of other monuments bear testimony to the greatness of our past civilization. The names of those who designed these magnificent monuments may be lost to us, but we cannot shut our eyes to the grandeur of their conceptions. Not only have they immortalised in this way the beauty of our ancient sculpture, architecture and painting, but in the carving of their caves and construction of their buildings, they have also exhibited a marvellous technical knowledge of the science of Geometry. For the caves of Ellora as well as those of Ajanta, and many old buildings like the Charminar, are noted for their fine mathematical proportions which add greatly to their charm and beauty and show the consummate art of the people who made them. It will thus be seen that a mathematical sense of proportion has a close relation to art. Today, in modern Hyderabad, Mathematics has been applied to vast works of irrigation and construction, while the architectural style of the Arts College building in which we are assembled shows that the ancient decorative art of the different peoples who inhabit this State is not quite lost to the present generation.

Mathematics, moreover, is the only science that is exact in the true sense of the word. It teaches us the great moral lesson of truth and exactitude; it is by mathematical principles that the mighty universe is governed which we see all around us, with its celestial bodies which, despite their stupendous size and grandeur, move in their stately orbits with a precision that is mathematical. It is Mathematics alone that reveals to us the mechanical secrets of the universe; and since truth is the essence of beauty, it may justly be said that Mathematics in its best sense is a pursuit or cult of truth and beauty which raises it to the pinnacle of the highest form of poetry.

Mathematics has for these reasons a great cultural value and must be allotted its rightful place in any scheme of education, the more so as its value in the sphere of mental discipline and training is incalculable. The Greeks knew this fact and Plato regarded this science as an indispensable factor in education and prescribed a course of mathematics as a subject of universal study. The famous inscription on the gate of the academy of the mathematician Euclid, the founder of Geometry, which read "Let no one enter who cannot geometrise" shows the importance attached to this science by the ancients. This was also the case in India which had the distinction in ancient times of being one of the most advanced countries in this science, especially in its relation to Astronomy.

On its purely practical side, Mathematics is a science of manifold utility. A knowledge of the subject is essential for the study of Physics and

Engineering, and mathematical methods are being increasingly applied to the elucidation of problems in purely lay subjects such as Economics, Logic and Psychology. There is, therefore, ample justification for calling it the "Queen of Sciences".

At one time it was thought that mathematics could not be made interesting or useful to a layman, and mathematicians of a past generation actually took pride in its supposed lack of utility. This is evident from the familiar story of the Cambridge Don, who said with pride regarding a theorem which he had discovered, that its beauty lay in the fact that it could never be put to any practical use! It is needless to say that such notions are now obsolete and as time goes on we find that even the most abstruse branches of mathematics are being increasingly applied to the natural and sociological sciences.

I hope the members of the Conference will be able to pay a visit to the Osmania Observatory which has been doing some original work in co-operation with other observatories and which possesses one of the largest telescopes in India It has had the good fortune to discover two new stars.

I am pleased to learn that you are devoting some time to a discussion of the methods of teaching mathematics. In this connection, the educational problem is to make it more attractive and interesting to children as well as to grown-up people. I expressly say this because the problem of adult education which is very much in the forefront at the present day, is just now absorbing the attention of the educational authorities.

While much work has been done more work still lies before you, for the possibilities of mathematics are almost infinite.

We in Hyderabad are deeply conscious of the honour you have done us in choosing this city and University for the present session of your Conference. I may say without making too great a claim that some of the members of the Faculty of Mathematics in this University have made an original contribution to mathematical science, more especially in connection with the theory of Einstein.

Let me, in conclusion, express my own good wishes and those of all of us for the success of the Conference. With your permission, I will now do myself the pleasure and honour of declaring this Conference open.

Address of Welcome

BY

THE CHAIRMAN OF THE RECEPTION COMMITTEE,

PROF. QAZI MOHAMMAD HUSAIN,

Pro-Vice-Chancellor, Osmania University

It is with feelings of utmost pleasure that I accord a most hearty welcome to all the members of the Indian Mathematical Conference, on behalf of the Local Reception Committee, to the historic City of Hyderabad and to the Osmania University. It is a very rare occasion that high mathematical talent of such a nature and magnitude could meet together in one place. Such

a gathering in Hyderabad is a matter of gratitude and happiness for all of us. The results of contacts between high intellectual workers in the field of a subject like Mathematics could be of inestimable value and of far-reaching significance as, apart from the mathematical discussions that will be carried out, the original papers that will be read and popular lectures that will be arranged under the auspices of the Conference here, a further great object will be achieved that mathematical minds pursuing the same lines of work and thought will be brought together from all over India, and lasting contacts will be made for the advancement of mathematical knowledge as the result of their co-operative efforts in their lines of study at the various universities and educational institutions in the country. It is hoped that your stay in this City, however short, will be pleasant to yourselves, and I am sure it will be full of benefit to Hyderabad and also it will be conducive to the making of such lasting contacts which I consider as one of the important aims of such intellectual gatherings.

We are all proud of ancient India's great heritage and achievements in the field of Mathematics. The world owes to India some of the basic ideas in the realm of Mathematics. The invention of the decimal system of notation in Arithmetic could be traced back to India, the foundations of Indeterminate analysis were chiefly laid here, and the vast knowledge of the theory and properties of numbers was generated in this country. India could be regarded as the birthplace of Arithmetic and Algebra which have in later times developed into modern analysis. The history of the subject of Mathematics confirms the conclusion that the Indian mind is highly responsive to deep mathematical thought. Eminent mathematicians like Brahma Gupta, Bhaskaracharya of old, and Srinivasa Ramanujam of modern age, not to mention any of the luminaries of the present company, were born on the Indian soil, under Indian conditions. It is a matter for pride to note that the most prominent ancient mathematician, Bhaskaracharya, was born in a village Bijjal Bida in Bid or Bidar District of the Hyderabad Dominions. New times have set in, and it is gratifying to observe that a keen type of scientist and mathematician is coming into the field, in good numbers. The mathematical teaching, as carried out in most places of higher learning in this country, is of a fairly satisfactory and advanced nature, although a great deal of headway has yet to be made in this direction.

The Indian Mathematical Society founded by Mr. Ramaswamy Iyer in 1907 for the advancement of mathematical study and research in India has done a great deal to advance the cause and study of Mathematics. Its achievements in the promotion of mathematical studies and the higher field of research over a period of about 30 years are really appreciable. Its two quarterly journals, one of them containing valuable original papers in the more advanced aspects of the subject and the other the " Mathematics Student," which meets the needs of young research workers and the teachers of Mathematics in schools and colleges, represent the valuable contributions to Mathematics of leading research workers and mathematicians in this country. It has built up a splendid library of mathematical literature and its most welcome feature is the appreciable number of high mathematical journals subscribed by the Society or received in exchange of its own journals. In order to promote the cause of Mathematics, it seems necessary that corporate organisations like the Mathematical

Conference and the Mathematical Society should be assisted by the individual efforts of all mathematicians in every possible way.

The Osmania University has only a short history. The subject of Mathematics is taught here in Urdu up to the M.A. and M.Sc. standard. The instruction of the subject has been all along in efficient hands, two senior members of the mathematics staff being Wranglers of Cambridge, and another senior member being also a high Honours man from Cambridge, with a Doctorate in Mathematics from the Leipzig University. The standards attained in the subject have been fairly satisfactory. The students of Mathematics from Osmania have gone to leading European and English Universities and have given good account of themselves, achieving a series of first classes in Mathematical, Engineering and other Science studies at such Universities. The Local Secretary, Dr. Raziuddin Siddiqi, is a product of Osmania who received his instruction in Mathematics in this department up to his B.A. degree stage. After that, he took his Honours Tripos in Cambridge in two years, instead of three, and later, on such foundations, he got his Doctorate in Mathematics from the Leipzig University. He is considered as one of the leading research workers in the subject in this country, and has produced several original papers which have been published in the leading journals of European countries and of India. He was awarded the gold medal by the Science Academy of Allahabad for the best research work carried out in Science and Mathematics during the last five years. Students of outside Universities—Benares, Aligarh, Madras, Calcutta and other places—are carrying on their researches under his direction from the Mathematics Department here.

The Nizamiah Observatory at Hyderabad, which is one of the valuable institutions of the Osmania University, has been doing excellent work in the domain of Astronomy under the direction of Mr. T. P. Bhaskaran. It has completed a number of volumes about astrographic work, relating to the various regions of the sky that were allotted by the International Astronomical Association to this Observatory. The whole work has been marked with precision and accuracy which is recognised by the highest authorities in the astronomical field. The work done at the Nizamiah Observatory has received recognition on international basis. Apart from the astrographic work, researches about double stars, occultations by the Moon, have been carried on with a telescope of 15″ aperture at the Observatory. Two more instruments are being added to the equipment of the Observatory, one of them a Spectrograph which will open up the regions of the sky with regard to the surface conditions, composition, classification and structure of stars as obtained from their spectra, and the other Spectro-helioscope which will facilitate the study of the surface conditions of the Sun. The students of Mathematics of this department have the unique advantage of paying frequent instructional visits to the Observatory and seeing the actual working of astronomical instruments in the Observatory.

The effective study of a subject cannot be divorced from the history of development of that subject. It may be suggested that the Indian Mathematical Conference and the Indian Mathematical Society may give lead in the matter and encourage the study of ancient Indian and Arabic literature of Mathematics. The Society could draw up a systematic plan and arrange to have it carried out successfully. The ancient important work could be

translated and published, with notes and commentaries. Such work, of course, could only be undertaken by competent scholars of the subject and of the languages concerned. Along with the creation and advance of the subject of Mathematics, it would be well to take stock of what has already been produced by ancient mathematicians of our country as their contributions to this great subject.

Having been in touch with the actual teaching of the subject to the University classes for some years now, it is my privilege to briefly lay before this Conference for their consideration, some observations and suggestions with regard to the work of Mathematics that is being carried on in our educational systems and universities.

The course of Mathematics for a student in schools and colleges consists of 14 years up to the degree stage—10 years at school and 4 years at the University—and if a student takes M.A. in Mathematics also, he devotes a further two years at the University. It may be asked whether as the result of 10 years work in mathematics, the standards reached at the school finishing stage are adequate enough. There seems a great deal of waste and repetition in the course of Mathematics that is done at schools. It may also be scrutinised if the years at the University, up to the degree stage, are being fully utilised and adequate standards produced as a result of four years' work in mathematics by the mathematics student. The same scrutiny may be continued for the M.A. degree course lasting over two years. It may be suggested that a body like the Indian Mathematical Conference will be doing a constructive piece of work, if they would draw up a syllabus for mathematics for the school, degree stage, and M.A. standard, on which a great deal of thought and collaboration of effort by competent members of this Conference may be available and suitable syllabuses, taking all factors into consideration, may be prepared for reference and guidance for the various educational institutions and universities. It is suggested that at the school stage much of the repetition and waste may be avoided and the student may be brought on to the use of the methods of calculus at the end of the course. At the degree stage, almost the same standards may be worked up as at the B.A. Honours degrees of English Universities. With regard to the M.A. stage, it may be considered that some association with research methods in the subject may form a part of the study of the student who takes the M.A. degree. It is now accepted as a principle in all Arts and Science subjects that M.A. and M.Sc. degrees should include the initiation of the student to preliminary research processes; but in Mathematics there is a reluctance, perhaps for sound reasons, to have it followed up. It seems a pity that standards all along the line may not be worked up in such a manner that a person, after 16 years' study of Mathematics, may not be brought into contact with some research methods with a view to enable him to advance the boundaries of knowledge to some extent if he follows up the study of the subject in life.

In order to bring about higher standards of the subject at the various stages, it is most essential that the teachers of Mathematics should have good grounding and control of the subject-matter that they teach. It is suggested that it should be the studied policy of Educational Departments to associate M.A. and M.Sc. trained people with the instruction of higher classes in schools.

Teachers should be given opportunities to join the universities and acquire the discipline of M.A. and M.Sc. studies, just as they are given facilities for training purposes in Training Colleges. Generally, it is considered that people who have taken their B.A. and B.Sc. degrees have adequate grounding to cope with the teaching of higher classes in schools. Being teachers of Mathematics, we know what a vast difference there is between the equipment of a B.A. and M.A. in Mathematics. Training in educational methods is, no doubt, a useful thing, although an opinion may be expressed that "teachers are born," but it may be stated that if a person has got no thorough foundations and grounding of a subject, a training cannot improve his stock of knowledge and abilities. It appears to me that the only effective way of raising the standards of subjects at schools, including Mathematics, is to associate good class M.As. and M.Scs. who have been trained in educational methods with the higher classes of schools. The experience of people would corroborate the fact as to what raising of standards was effected in a school where such men were placed in charge of the instruction of such classes.

With a view to raise the standard of subjects at the university stage, we realise that the functions of a university on the academic side are two,—instruction and research. With regard to the former, it is essential that a university teacher may make himself fully conversant with the "known" knowledge about the branches of the subject that he teaches, by a wide intensive study which gives him the complete mastery and control over the subject that he teaches. With regard to the second, it may be stated that all members of the teaching staff in a university, should be engaged in some kind of original work and research relating to their lines of study. In the hands of a teacher absorbed in the original research sides of a subject, the study of a subject acquires a freshness, power and inspiration that seldom comes to the teaching of one who simply lectures and offers summaries of books as lessons to students. As a matter of fact, "No person can justify his position as a member of the teaching staff of a University, unless he is engaged in some kind of research himself." It is also suggested that the members of the university teaching staff may acquire working knowledge of German, French and other European languages which will help them with regard to their instruction and research in the subject. It may be the avowed aim of universities to expect such grounding of the subjects from all members who join the staff of universities as teachers of higher classes. It may be added that such expectations are made from the members of the staff in this University.

It would be advisable for universities to co-operate in establishing a system of higher extension lectures in Mathematics which may not be entirely research lectures, but may be something of post-M. A. standard, relating to the most advanced, known sides of Mathematics. Lectures on Quantum Mechanics, Mathematical Physics, Relativity Mathematics have been started during the last two or three years at this University, and intimations were sent to all the universities in case any of their scholars or members of the staff would like to take advantage of them. It is suggested that as a corporate effort, such higher lectures may be arranged at suitable times which could conveniently be availed of by advanced students and members of the staff of other places.

These are the days of rapid progress and new knowledge is being added almost every day. It is impossible for any worker, however brilliant he may be, to master all the branches of the subject. Specialisation on firm and comprehensive foundations has to be encouraged. For this purpose, young students and research workers should have ample opportunities of coming into contact with distinguished mathematicians who are working in similar fields, so that they may get new ideas and outlook in the subjects of their interest. Such an object could be achieved by the periodical exchange of Professors or it could be done by introducing the migration system for advanced students with the object of allowing them to study at different centres as is done at most of the continental universities.

The research side of the subject, takes no notice of the fact that researches are being conducted in Applied or Pure Mathematics. They should be carried on as flashes in the mathematical field occur to the worker. With regard to the known stock of knowledge of the subject, however, it is suggested that a special effort may be made to apply this knowledge to the conditions and things round about us in nature. A special aim, along with others, of the mathematical workers may be to develop the sides of Applied Mathematics, in its applications to a wider field in sciences and other subjects. Aeronautics is a coming-up science and there have been enormous developments in the field of this science during the last decade or so. Before long it may come about, that most of the world communications will be through the air. Railways and steamships will be thrown to a relative place in the background and there may be no need to constantly maintain railway tracks, and establish huge harbours and highways on a large scale. India is a vast continent having special facilities for aeronautic travel. Aerodynamics and the application of Mathematics to aeronautical problems may be studied with great advantage as a part of Mathematics,

Mathematics is an abstract subject and is taught in a purely abstract manner. Perhaps it is the great merit relating to this subject, that mathematicians receive a lifelong training in dealing with the abstract conceptions of things, and have their imaginations developed in an enormous degree relating to the subtle concepts of the physical universe. It will, however, be a help to the mathematical thinker if the "concretes" of Mathematics are presented to him in the class room whenever possible, with a view to relieve him of the strenuous strain of the intense abstract thought. For example, in the teaching of Applied Mathematics, demonstrative work with balls, inclined planes relating to rigid dynamics may be supplied in the class room. With regard to higher surfaces in geometry, concrete models will help a great deal, and in wave motion study, various illustrations may be used. In Astronomy which is taught as a part of the Mathematics course, copious use may be made of black spheres and celestial globes apart from the handling of instruments by students with regard to the determination of astronomical data. It is suggested that suitable laboratories may be built up along with the teaching of mathematical subjects and the concrete aspects of the subject may be imported from the Science of Physics, Engineering, etc., and other illustrative appliances and models may be made in the mathematical workshop and laboratory.

It has been pointed out that the chief feature of Mathematics is that it is an abstract science. It represents a discipline for concentration and regulation

of thought that few other subjects could afford. A student of Mathematics of the lower and higher grades is always solving problems, in fact making discoveries with regard to the phenomenon of life at all moments of his work. A mathematician sees things in flashes, He tries to discover the causes why certain things have happened and he is always trying to build up conclusions from sets of facts that constitute the data of a particular problem. He carries on these processes of thought all his life, with most complicated data, with the result that the simpler facts and relations of life come easy to him. He has to turn his attention to any phase of life and he will see through deeper into it understand things more out of it than perhaps a person who has not had such training. With regard to all great undertakings and organisations which are launched on enormous scales by individuals or nations, there are two essential aspects—the imaginative trained thought that conceives all the detailed aspects of a programme, the other is the power of executing the programme Mathematics will remain as a most vital discipline with regard to the former and the ability to carry out the programme depends on the personal qualities and character of a man who is placed in charge. It has been made out by various eminent writers that a mathematician understands the world all round better in view of his superior discipline of thought. It may, however, be pointed out that his data is oftentimes meagre and inadequate. It will be advisable for him to go about much, observe much, and increase his experience of life by all possible means.

In the end I may say that this is an assembly of seekers after mathematical truths,—a type of truth that is unperishable through all time and is above all personal and world feelings. It is a type of truth that is nearer perfection. It is my signal privilege today to welcome such seekers of truth to the land of Hyderabad and to the Osmania University. I assure all the delegates of the Mathematical Conference that the heart of Hyderabad has welcomed them here.

Secretary's Report for the period March 1938 to December 1939

BY

DR. RAM BEHARI, *Honorary Secretary.*

MR. PRESIDENT, LADIES AND GENTLEMEN :

1. On behalf of the Managing Committee of the Indian Mathematical Society, it is my pleasant duty to extend to you all a hearty welcome to this Conference. We are profoundly thankful to the members of our Society and to other distinguished mathematicians who have undertaken long and tedious journeys to attend this Eleventh Conference of our Society. We are grateful to the many citizens of this famous city who have responded to our invitation and are gracing this occasion with their presence.

We are specially grateful to you, Sir, for the trouble you have taken, in spite of your multifarious duties to come to inaugurate our deliberations.

2. Our Society was founded by the late V. Ramaswamy Aiyar and a group of enthusiastic associates in 1907, and since then it has been growing steadily in strength and usefulness under the wise guidance of its successive presidents.

During the time that has elapsed since we last met at Lucknow in March 1938, 22 new members have been enlisted including 2 life-members.

We are proud to count as one of these new members the eminent mathematician, Professor N. R. Sen who is to deliver the Inaugural Address at this conference.

I regret to have to record the loss which the Society has sustained in the death of Pandit Hem Raj, M.A., Principal and Professor of Mathematics, Doyal Singh College, Lahore, who was a very old and valued member of our Society and an active member of the Managing Committee for over 13 years.

3. Our Society has three main activities viz., holding the biennial Conferences, publication of the two Journals and maintaining a library of advanced books and journals.

The first conference of the Society was held at Madras in December 1916. Since then we have had conferences at Bombay, Lahore, Poona, Bangalore, Nagpur, Trivandrum, Delhi and Lucknow. We are now holding our Eleventh Conference in this unique centre of learning under the auspices of the Osmania University which has given the much desired lead to the country by imparting instruction in all subjects including mathematics, through the mother tongue, up to the highest standard.

It will serve to show the interest evinced by our members in these periodical conferences, if I mention that in spite of the anxious times through which our country is passing on account of the European War, and the inconvenience that might have been caused by the temporary postponement of the Conference, our members and well-wishers have assembled here in large numbers to make this Conference a success by their presence and mathematical contributions. The number of papers that have been communicated to this Conference is nearly 60 as compared to 40 at the last conference. We have also been able to arrange symposia on modern research topics, and a discussion on the courses, and methods of teaching mathematics with a view to bring about a higher standard of mathematical instruction in the country.

4. The Society continues to publish two journals, each of which appears four times a year. The first, viz. '*The Journal of the Indian Mathematical Society*' is edited by Dr. R. Vaidyanathaswami of the Madras University and publishes only original papers and caters for the needs of workers in the field of advanced mathematics. The other one, under the title, '*The Mathematics Student*' is edited by Professor A. Narasinga Rao of the Annamalai University and serves as a link between higher mathematics and the sphere of collegiate mathematics, and also publishes 'News Items' with the object of keeping the readers in touch with one another and with the leading events in the mathematical world in India and abroad.

Besides the journals, the Society has published this year with the financial aid of the Punjab University Dr. Hansraj Gupta's 'Tables of Partitions' a book

which it is hoped will be of great value to workers in the Theory of Numbers.

5. The Library of the Society continues to be under the charge of Professor R. P. Shintre and Professor D. D. Kosambi, of the Fergusson College, Poona. The Library took over the whole of the exchange department, and consequently the journals sent in exchange were mailed from Poona. But before this new scheme could get well under way, the war broke out, and at present, three separate issues of our publications await the chief Censor's permit to be mailed to non-allied countries. On the other hand most of the journals that come to us from abroad are received even now except Journals published in Germany, Czechoslovakia, Poland and journals like 'Science Progress' in England, the publication of which has been suspended.

The circulation work continues as before, but members are forewarned that the war is likely to interrupt our service to some extent.

6. With regard to the financial position of the Society, every year several members of longstanding are taking advantage of the reduced rate of life composition to become life members, and most of the new members are enlisted on the concessional rate of subscription, with the result that the annual income from ordinary membership fees is gradually dwindling. Most of the income got as subscriptions and grants-in-aid from Universities is spent in the course of the ordinary routine work of the Society, and hardly any surplus is left for increasing the efficiency of our service.

The Managing Committee hopes that with the willing co-operation of our members, the financial position will improve by enlisting new members and institutional subscribers for the journals, and that more Universities, Provincial Governments and Ruling Princes will come to our aid by making handsome annual grants.

7. In conclusion, I take this opportunity of giving expression to the thanks of the Society to the Universities that have rendered financial help in the past by giving annual grants-in-aid, and to the various local Governments, Indian States, Universities and other institutions for the facilities they have granted to their officers to enable them to attend this Conference.

I have also great pleasure in expressing, on behalf of the Managing Committee, our thanks to the Vice-Chancellor of Osmania University for giving us this opportunity of assembling here by his kind invitation, and to His Exalted Highness, the Nizam who has been graciously pleased to send a message of encouragement.

Our special thanks are due to Dr. M. R. Siddiqi, his band of volunteers, and other members of the Local Reception Committee for their warm hospitality and excellent arrangements for the comfort of the guests.

Inaugural Address

BY

PROF. N. R. SEN,

Ghosh Professor of Applied Mathematics, Calcutta University

Mathematical and Physical Science

Before proceeding with my address I must express my sincere thanks to you for the honour you have done me by inviting me to deliver this inaugural address, and also state how deeply I appreciate this kindness. I am conscious of my limitations, as I am interested in subjects which may not appeal to mathematicians; nevertheless I will try my best to discharge faithfully the function which I have been called upon to perform.

A well-known physicist of this country once complained to me saying that Mathematics was leading physicists more and more into absurdities. This coming from one who himself has made no mean contribution in his subject is indeed a serious charge. I shall take this opportunity to discuss the question of the relation of Mathematics to the physical sciences, and examine if the latter have really suffered by contamination from Mathematics.

To the common people Mathematics is a subject inspiring awe and reverence. Usually school children dread it, students in Colleges and Universities avoid it, and the average man of the world regards it as a subject too high and difficult for comprehension, though not infrequently in India at least, he is found to take part in discussions in abstruse philosophy with pleasure and enthusiasm. This reputation of Mathematics has no doubt its origin in the abstract nature of the subject, but this is certainly not the whole truth. The reason for the reverence for Mathematics is the common impression regarding the precision and infallibility of its calculation. A subject creates great confidence if its conclusions are arrived at by mathematical processes. It is the application of Mathematics to physical and engineering sciences with their wonderful achievements contributing so much to the development of civilisation, which is the real secret of its success from the ordinary man's point of view. Mathematics is regarded by him as a scientific apparatus by which it is possible to make predictions. Whether it be the question of the amount of electrical energy which will be available from a falling stream of water, or the strength of the walls of a boiler necessary to withstand the pressure of steam, or a premium table which will neither cause an Insurance Company to collapse, nor put severe burden on the shoulders of the participants, in every case it is this powerful weapon of Mathematics which is put into operation to predict results which do not fail. In short the average man's attitude towards Mathematics is determined by the use of Mathematics for solutions of the practical problems of the Applied Sciences, and to the extent they have given him immense advantages in life, an amount of infallibility is attached to mathematical calculations.

It will probably surprise the average man to be told that mathematicians have quite a different idea about their own business. To them Mathematics is not a Science. The methods of observation, experimentation and induction by

which science gains knowledge and advances are unknown in Mathematics. A mathematical truth differs in its very nature from the truth of a scientific discovery. While Science considers reality to lie only in the external world, whose laws are to be deduced mainly by observation and induction, mathematical truths are independent of such considerations. There is thus a great gap between the attitude of the average man and that of the mathematician towards Mathematics. Here one may raise the question of the relation of mathematical truth to the external world, of how and in what manner the processes of nature find a representation within the framework of mathematical systems. An answer to this question can truly interpret the two different attitudes, as well as clearly define the limitations of such a representation.

In technical sciences involving the use of mathematics, mathematical rules and formulae are abundant. The ordinary computer uses formulae often quite mechanically; the expert works out his own formulae under the conditions of his problem. Barring purely empirical results, there is the general tendency to correlate technician's formulae to well-known mechanical or physical laws. The better a formula is in consonance with such a law, the more is the satisfaction of the technician. In ultimate analysis technician's mathematics turns out to be nothing else than an attempt to represent physical laws by mathematical formulae. The mathematical processes of the technicians are well known to be approximations. Klein calls it "approximation mathematics", as distinguished from the "precision mathematics", of the abstract theory. In essence the two are not incompatible, the former is only a phase of the latter, which is a beautiful ever-expanding edifice in an ideal world, and will survive even if the real world crumbles and passes away. We do not propose to go into a detailed discussion of Kleins classification on this occasion. It is enough for us to notice here that engineer's mathematics is, so to say, a sort of calculating machine the inner wheels of which move according to some known, or as yet unknown laws of nature.

In recent years there has been much discussion on the question of what are laws of Nature, and how knowledge of these laws is derived. The attitude of modern science is to recognise knowledge only when it comes through experience, and deductive and inductive processes of thought. The investigator gets his primary knowledge by observations, experiments and measurements. By this process he knows only isolated facts. One can prepare in this manner a catalogue of facts from observations of natural phenomena; this catalogue will indeed be infinitely long. To the primary man, if we think he had not the power to reason, Nature must have revealed herself in this chaotic aspect. Such a view is repugnant to the aesthetic sense of the civilised man. A spiritual need impels him to look for order and harmony in this chaos. This is the origin of his belief that there are laws in Nature. Then arises the need for studying his catalogue closely, rearranging and grouping the data guided by intuition. There appear correlated strings of facts which combine into forms and principles from which by Induction a law of Nature is derived. The process of this derivation in science must be a slow and cautious one. In strictness the truth of such a law can be admitted only in the environments in which the experiences of the investigator lie. Nevertheless, extrapolation of such laws to totally foreign environments, as a tentative measure, is a common tactic adopted by Science in her

endeavour to extend the bounds of her dominion. Thus the receptacle of what we call physical reality is Nature. The laws are the outcome of an urge in intelligent beings to find an order in Nature. It is not the discovery of this order alone that brings full satisfaction of this urge. The mind of Man works on a more ambitious line, and demands such a knowledge of the structure of the Nature as will enable him to make predictions. There thus arises the necessity for constructing an apparatus by which starting from a set of known facts, a different and as yet unknown set may be predicted. Where is the apparatus which will make this wonderful achievement possible? Human intelligence has fashioned it not out of materials taken from Nature, but from imagination working in a totally ideal plane.

While we cannot conceive of physical reality apart from experience, Mathematics has a realm apart from human life and experience. We may say the "reality" in Mathematics does not lie in the world of our experiences. Here we start from certain conceptions, axioms and postulates which need not have any connection with the facts of the Nature. The essential point is that these axioms and postulates should be free from inner contradiction, and eventually form an independent and complete set. On this basement the entire structure of the mathematical system is raised by deductive methods. All the beautiful and elegant theorems known in different branches of *Mathematics have their roots deep down in certain sets of fundamental* axioms and postulates. Even the finest achievements of abstract Mathematics are, from this point of view mere "tautological transformations". The only principles that govern these transformations are the principles of deductive Logic. Euclidean Geometry for example, is a strictly logical system of this kind based on some axioms and postulates. Its conclusions are exact, and strictly accurate. Its postulates require that through a given point outside a given straight line there should be only one straight line parallel to the given line. This indeed agrees with our usual experience. Bolyai later showed that one can construct a geometrical system in which more than one such parallel may be drawn through the point. In this system the three angles of a triangle are less than two right angles; still the system is logically exact just like the Euclidean system. Riemann followed with a Geometry in which no parallel can be drawn through the point. Geometry thus builds only logical systems the contents of which may look quite absurd from the point of view of our experiences of the world. Mathematical objects have not necessarily any concrete existence. The mathematician starts with objects A, B, C, they may be stars or sticks, for him both are equally good. He is concerned only with the relations among the objects, not in the objects themselves. The constructions in Mathematics are only logical systems which are entirely empty in as much as they do not contain any physical reality. This view indeed has not been shared by all mathematical philosophers, but it appears to be the one which is more and more receiving a general support from the side of mathematicians as well as from that of the scientists.

If we admit this to be the real character of abstract Mathematics, we have to find a nexus between the world of facts, and the world of empty logical systems. The latter containing strings of deductions from a set of primary postulates are admirably suited to supply the fundamental need of a predicting machinery about which we have spoken before. Take for instance a differen-

tial equation. It gives us definite information about the nature of the dependence of one variable on another, when we know something about this dependence in isolated cases. This is an apparatus which is extremely handy to reproduce the sequence of a string of facts which may be considered to possess causal links. Take the dynamical problem of a planet in motion in the gravitational field of the Sun. Our inquiry about the motion ultimately reduces to this: if we know the characteristics of the motion of the planet at any time t, how can we calculate the motion at any subsequent time? If the motion be represented by a differential equation, this will give us information about the motion at the instant $t+dt$ when that at the instant t is known. Such a predicting machinery, if it can reproduce facts in the requisite order is nothing else than what we call a law of Nature. All differential laws represent in this manner either the evolution of a physical system in time following what we call a causal chain, or the connection between the infinitely close elements of the system. This is the essence of all Field Theories. But a differential equation is not identical with a Field-law. Its entire content is the peculiar dependence of a varying magnitude on another (no matter what these magnitudes refer to) independently of whether it happens to embody a physical law or not.

Also the modern atomic theory built on the conceptions of certain discontinuous processes clearly shows the same aspect. In classical mechanics conceptions like positional co-ordinate, momentum, and energy are directly representable by ordinary mathematical quantities which are subject to usual algebraic operations. In the domain of atomic mechanics these variables, in order to give correct representation of facts, must be considered to obey a non-commutative algebra. After a trial of over ten years the facts of the atomic motions and transformations were admirably fitted to the algebra of matrices, and the theory of their transformations. The transformation law is a differential one, so that the modern atomic theory has a complicated structure to which the continuous system of partial differential equations, and the discontinuous system of non-commutative algebra have contributed their shares.

The association of Mathematics with the Physical Sciences thus appears as a process of fit. We have seen Mathematics is concerned with the construction of logical systems of different types. But the possibilities here are infinite. These possibilities indeed may be existent only in ideal worlds which are not built after the models of the world of our experiences. Our knowledge of the facts of the Nature is the direct result of our experiences gained by observation and experimentation. Our attempt to find order in these apparently incoherent masses of facts and discover the Laws of Nature leads us to seek which of the known mathematical systems will not only describe these facts correctly, but by a deductive process of thought inherent in the system predict new results. The choice of the system is only a question of fit and convenience. Science cannot deny the possibility that there may be more than one system which may suit her as regards this representation. The scientist will undoubtedly choose the simplest amongst them. Mathematics supplies a steel-frame of construction into which the scientist puts his materials of facts and raises a fine edifice satisfying his inner aesthetic sense of order and beauty.

In the association of fac's with logical systems lies the origin of what is known as approximation mathematics of the Applied Sciences. All observations of man are subject to errors, as all his measuring instruments have limitations. On the other hand the mathematical systems are all strictly accurate in the sense that they are logical deductions from certain postulates. Euclid's geometry is indispensable to the engineer in his constructions, but none of his measurements will *exactly* conform to Euclidean geometry. Again take for example, the well-known principle of Energy which the physicist still believes to be inviolable. In the progress of physical science it has made and is still making invaluable contributions. But its verification is never so exact as the mathematical formulation of the principle requires. The principle can thus be taken to be strictly satisfied only in an ideal world, while in the real world of our experiences such deviations are possible as may be correlated to the errors of observation and measurement. Thus the physicist's or engineer's need is a method of calculation which has bearing only on the measurements of the phenomena of the real world. Anything that does not come within the reach of his measuring instruments does not simply interest him. He has no need of irrational quantities, as his measurements do not go beyond a few places of decimals. Nevertheless all his results correspond to certain perfect specifications of an ideal world which is the creation of Mathematics.

The question of the method of choice of a suitable mathematical system is one to which a satisfactory answer is yet to be given. The latitude of an arbitrary choice cannot naturally appeal to one who has set himself the task of constructing a world of harmony and order. As in abstract Mathematics we can construct axiomatic physical theories applicable only in ideal worlds different from our own. For instance, one can imagine a gravitational system in which the law of force is the inverse cube of the distance, instead of the square. This system will have a dynamics different from ours. One may go even further and imagine a world where apples instead of falling down from trees go upwards, where a magnet repels a piece of iron, or where an atom has positrons moving round a heavy electron. Infinities of such systems are imaginable, in which things happen in contradiction with the experiences of the real world. We may call them worlds of miracles in preference to the term "unreal systems". How is the system in which we live to be distinguished from such "unreal systems", in other words how is the real world to be differentiated from these worlds of miracles? We have to face this question when we proceed to exercise our choice of a system for the representation of the facts of Nature. There has been a general tendency in the different branches of Physical Science to use a definite test for this purpose. It is the so-called Action principle, according to which the natural system is characterised by the property, that in it the integral of a certain invariant function is a minimum. Consider a closed space in which you are looking for a law of Nature. Inside of this you start by assuming the possibility of the existence of any one of the worlds of miracles, or of the real world. A mathematical function called the Action function which will have the same value for all observers in any kind of motion, and of which the construction does not depend on a particular law of Nature, may be built and its integral for the whole space concerned taken into consideration. This space which we shall describe by the

word " Field " will be characterised by certain quantities. In fact with every point of the Field will be associated quantities of this kind known as field variables. This association describes a Field which may correspond to the real world, or a world of miracles. Now a variation of the integral of Action will involve a redistribution of the value of the field-quantities, that is passing in general from one world of miracles to another. In this variation the boundary of the Field must not take part; there the conditions existent in the real world are held fixed. The Action Principle states that of all distributions of the field quantities that of the real world makes the integral minimum or strictly speaking an extremum. If then, according to a well-known mathematical principle you change the field-quantities very slightly, that is, pass from the real world to a world of miracles only slightly different from it, the Action integral, to a first approximation will remain unchanged. This condition implies a sort of coherence in the hitherto amorphous Field, and the law of this coherence is the law of Nature. But though a variation principle of this nature is recognized in most branches of physical science, its real import has not been brought to light. We note it is only here in the whole domain of Science that there is a comparison between the natural system on the one hand, and the entire manifold of non-natural systems or worlds of miracles on the other. The natural system possesses a distinct stationary characteristic. It satisfies our aesthetic taste to think that among the infinity of possible systems there is one which has a unique property, this being the natural system. But this uniqueness is bound up with particular forms of the invariant Action function. Why this should be so is a mystery of Nature which as yet remains unrevealed.

In the domain of atomic physics a different test is applied to restrict the choice of systems. In interpreting atomic phenomena in terms of electronic motions inside the atom, it so happens that these motions can be associated with certain sets of integral and half integral numbers called quantum numbers. A particular association of these quantum numbers determines a state of the atom. Among all possible electronic motions within the atom, only the motion in the natural system satisfies the Exclusion Principle of Pauli, which allows one and only one electron in a completely defined quantum state. One can certainly think of other models of atoms but they do not populate the world of our experience. In the case of discontinuous atomic phenomena the selection of a suitable mathematical system is thus more direct than in the Field theories.

These considerations show how Mathematics and Physical Sciences originate, and how they develop along their own lines. In the progress of Science Mathematics is bound to play a prominent role. The exact laws of Nature are expressed as mathematical equations. The conclusions drawn from these equations are again interpreted in terms of natural phenomena and often lead to new knowledge. Yet physical laws do not serve any mathematical interest nor are mathematical systems primarily designed to represent physical laws. The domain of Mathematics is closed and selfcontained and is independent of physical realities. If it leads you to an unreal world, it is due to your own choice.

ABSTRACTS OF PAPERS

M. Venkatarama Ayyar and M. Bhimasena Rao, Bangalore.

On types of solutions of $x^3+y^3+z^3=1$ in integers.

S. Ramanujan had given this problem as Question 681 in the J. I. M. S. for 1915, adding 6 illustrative examples of which the last was $67402^3-83802^3+65601^3=1$. A partial solution of this was given by Prof. N. B. Mitra in the J. I. M. S. for 1921, verifying only the first two cases and omitting the rest. In the present paper, the authors have brought out the underlying unity among all the examples, which come under type 4, the type being defined by the value of $(1-z)/(x+y)$. The same method is extended to obtain examples of other types and complete solutions for types 3, 4, 7 and 13 are furnished. As simple examples may be mentioned,

under type 3,	$3753^3-2676^3-3230^3=1.$
	$3753^3-5262^3+4528^3=1.$
under type 7,	$94^3-103^3+64^3-1.$
under type 13,	$8343^3-8657^3+4083^3=1.$ etc.

The formula given by Ramanujan in Q. 441 (J. I. M. S. 1913),

$$(6a^2-4ab+4b^2)^3-(4a^2-4ab+6b^2)^3-(3a^2+5ab-5b^2)^3=(5a^2-5ab-3b^2)^3$$

is of our type 4 and is a variation of a formula given by J. R. Young much earlier,

$$(2d^2+4d+42)^3-(2d^2-4d+42)^3+(d^2-16d-21)^3=(d^2+16d-21)^3$$

Similar formulae for types 3, 7, 13, 19 etc. are also obtained.

D. R. Kaprekar, Devlali.

Demlofication of Numbers in Arithmetic Progression.

Take an A. P. say $n, n+d, n+2d, \cdots$ and write the terms one below the other advancing the unit digit one step to the left each

time. When the numbers are added we have a group of recurring digits. Thus for $n=17$, $d=6$ the recurring group is 814. In this paper it is pointed out that the repeating group depends on d and that there are 4 cases.

$$\begin{array}{r}
17 \\
23 \\
29 \\
35 \\
41 \\
47 \\
53 \\
59 \\
65\\
\hline
\cdot\cdot 1\,4\,8\,1\,4\,8\,1\,4\,7
\end{array}$$

Case (*i*) If d is a multiple of 9 the group consists of 1 digit only.

Case (*ii*) If d is a multiple of 3 but not of 9 or 10, the group consists of 3 digits which may be any one of the following or a cyclic permutation of the same

074; 185; 296; 370; 481; 592

Case (*iii*) If d is any number not divisible by 3 or 10 the repeating group consists of 9 digits which may be any cyclic permutation of 6 groups of 9 digits.

Case (*iv*) Covers the case of d being a multiple of 10.

S. S. Pillai

On Waring's Problem

Dickson has proved that $g(5) \leqslant 55$ and $g(4) \leqslant 35$. In this paper I show that $g(5) \leqslant 41$ and $g(4) \leqslant 27$.

K. Sambasiva Rao, Andhra University, Waltair.

On Waring's Problem for Fifth Powers.

Let G(k) denote the least integer s such that the Diophantine equation

$$N = x_1^k + \cdot \cdot + x_s^k,$$

is solvable in positive integers for all sufficiently large integers N.

It was proved by Hardy and Littlewood that $G(5) \leqslant 41$. Subsequently several writers improved it: James to $G(5) \leqslant 35$; Estermann to $G(5) \leqslant 29$; Hua to $G(5) \leqslant 28$.

The purpose of the present paper is to prove that

$$G(5) \leqslant 25.$$

The author has been able to arrive at this result by improving, among other things, a theorem of Davenport on 'admissible exponents'. Other consequences of the method are

$$G(6) \leqslant 40; \quad G(7) \leqslant 56$$

These are improvements on the previous results.

Hansraj Gupta, Hoshiarpur.

Waring's Problem for powers of primes—II.

In a recent paper Dr. S. S. Pillai has examined one of my conjectures and proved the theorem:—If $l \leqslant r \leqslant 2^n - 2l - 2$ and $n > 20$, then every integer not exceeding N is the sum of at most I nth powers of primes $\geqslant 1$ where

$$N = e^{(1 \cdot 6)^n/6} \cdot 31^n,$$
$$3^n = l \cdot 2^n + r$$

and
$$I = 2^n + l - 2$$

He has not stated how far his theorem holds for values of $n < 20$. The object of the present paper is to study the validity of Pillai's Theorem in this case and strengthen the following conjecture: If $n \geqslant 7$, then every integer can be expressed as the sum of at most I nth powers of primes $\geqslant 1$. For $n < 7$ this conjecture is easily seen to be false.

V. Narasimhamurti, Waltair.

On a Problem in Arrangements.

PROBLEM.—"$2n+1$ people are invited out to dinner on n different days. Is it possible to arrange them round a circular table in such a way that no person has the same neighbour on different days?"

Suppose the following is a solution,

1st row $(y_{n+1}+n,\ y_n+n,\ \cdots\cdots,\ y_2+n,\ y_1,\ y_2\ \cdots\cdots,\ y_n,\ y_{n+1})$

2nd row $(y_{n+1}+n+1,\ y_n+n+1,\ \cdots\cdots,\ y_2+n+1,\ y_1,\ y_2+1,\ \cdots\cdots,$

$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \qquad y_n+1,\ y_{n+1}+1)$

nth row $(y_{n+1}+n+n-1,\ y_n+n+n-1,\ \cdots\cdots,\ y_2+n+n-1,\ y_1,$

$y_2+n-1,\ \cdots\cdots,\ y_n+n-1,\ y_{n+1}+n-1)$

where $y_1 = 1,\ y_2 = 2,\ y_3 = 2+x_1,\ y_4 = 2+x_1+x_2,\ \cdots\cdots$

$y_{n+1} = 2+x_1+x_2+\ \cdots\cdots\ +x_{n-1}$;

and the x's are all among the numbers $1, 2, 3, \cdots\cdots, n-2, n-1, n+1, n+2, \cdots\cdots, 2n-1$.

It should be noted that any number —(excepting the number $y_1 = 1$)—which is congruent to r (mod $2n$) is to be replaced by r if $2 \leqslant r \leqslant 2n-1$; and by $2n+1$ or $2n$ if $r=1$ or 0 respectively. We have thus all the $2n+1$ numbers from 1 up to $2n+1$.

Now the solution will be possible if all the x's satisfy the following two conditions,

(1) $x_r + x_s \neq 2n$ for any r and s

(2) $x_m + x_{m+1} + \cdots\cdots + x_{m+t}$ is not divisible by n,

For, the second ensures that the numbers in each row are different, and the first excludes the possibility of any two neighbours in one row appearing again as neighbours in another row.

The above conditions are satisfied if we choose $x_1=1$, $x_2=2n-2$, $x_3=3$, $x_4=2n-4$, $\cdot\cdot\cdot\cdot$ $x_{n-1}=n-1$ or $n+1$ according as n is even or odd.

S. M. Kerawala, Aligarh.

The Problem of the Play of Thirteen.

The problem is generalised in this paper for three players in the form: Three players A, B, C are given three well-mixed packs, each consisting of the same n different cards. They expose together singly cards out of their respective packs. What is the probability that no two of any set of three cards exposed together be the same? If u_n is the probability, it is shown that u_n satisfies the difference equation

$$u_{n+5}=\frac{(n+4)(n^2+8n+17)}{(n+3)(n+5)^2}u_{(n+4)}+\frac{(n^2+8n+17)}{(n+4)(n+5)^2}u_{(n+3)}$$
$$+\frac{(n^2+8n+13)}{(n+3)^2(n+4)(n+5)^2}u_{n+2}+\frac{2(n^2+5n+3)}{(n+2)(n+3)^2(n+4)(n+5)^2}u_{n+1}$$
$$-\frac{4u_n}{(n+5)^2(n+3)^2(n+2)(n+1)}.$$

B. S Madhava Rao, Bangalore

On the limits of the roots of a polynomial equation

A simple proof is given of a theorem due to Laguerre on the limits of the roots of a polynomial equation all of whose roots are real and distinct. Further theorems are proved which sharpen these limits.

K. V. Iyengar, Bangalore.

A deepening of the Binomial inequality.

We know that $E=\dfrac{x^n-1}{x-1}$ lies in (nx^{n-1}, n). This inequality is deepened by elementary methods so that depending on the value of the exponent n, $(1<n<2$, etc.) we can determine in which of the subintervals formed by marking the A. M. G. M. and H. M. of the extremes of the above interval, E lies. A particular case of the same where n is an integer is given as a problem in Hardy-Littlewood-Polya, *Inequalities.*

F. W. Levi, Calcutta.

Pairs of inverse moduls.

Two submoduls A and A′ of a (commutative) field F will be said to be *inverse* if the elements different from zero of A are inverse to those of A′. If especially $A = A'$, the modul is *self-inverse*. If the characteristic of F is different from 2, one gets all the pairs of inverse submoduls by

$$A = aK, \quad A' = a^{-1}K,$$

where K is a subfield and a is an element of F. If in particular a is an element of K, then $A = A' = K$; if a^2 (but not a) is an element of K, then $A = A'$ is self-inverse, but is not a field. If a^2 is not an element of K, then A and A′ have no common element besides zero. The case of characteristic 2, needs a special discussion which is also given in the paper. Finally the theory is applied to the geometry of the Euclidean plane.

R. Vaidyanathaswamy, Madras.

Characterisation of the Quasi-Boolean Algebra as a Distributive Lattice.

The result is proved that the necessary and sufficient condition for a distributive lattice to be a Quasi-boolean Algebra is that every ideal containing a prime ideal be itself prime.

T. Venkatarayudu, Masulipatam.

On the linear Algebra of classes of elements in a finite Abelian group.

In my previous paper under the same title I defined $C(d)$ as the class of elements of order d in a finite Abelian group G and showed that the classes $C(d)$ combine among themselves by the group operation. Let the given Abelian group G be of order N and let d be a divisor of N. We denote by $B(d)$ the totality of elements in G which can be expressed as the dth powers of elements in G but no higher power d' of any element in G, d and d' being divisors of N. The object of the present paper is to show that the classes $B(d)$ combine among themselves by the group operation. This algebra defined by the classes $B(d)$ will be found to be quite analogous to the algebra of the classes $C(d)$. The two algebras will be identical if and only if the given group is cyclic in which case $C(d) = B(N/d)$.

M. Zia ud Din, Aligarh.

Groups and Conjugate Matrix solutions of equations.

Recently Groups and Matrices have attained a prominent place in modern Physics and there is a wide field of research in them. In this Paper the Characters of the Symmetric group have been used to obtain the Conjugate Matrix solutions of certain equations. A general method is indicated.

K. S. K. Iyengar and K. V. Iyengar, Bangalore.

The solution of an Extremum problem in interpolation-theory.

In this paper we solve the following problem. Let $y=f(x)$, be such that $\bar{\phi} > D^n(y) > \phi$, where $D^n(y)$ exists at all points except at a set of measure zero, $D^{n-1}(y)$ being continuous throughout a given interval (a, b), and $\bar{\phi}$ and ϕ are summable. Suppose further that the values of y and its derivates up to $(n-1)$th orders are given at a and b. The first problem that arises is to find the conditions to be satisfied by the given extreme values in order that a y with the imposed conditions exist. The second problem is to find the upper and lower curves in case the values are consistent. These two problems are solved in this paper. The principal difficulty is to establish the existence of the extreme curves, which has been done by a somewhat unique method which is perhaps capable of being applied to a larger class of problems. We have generalised this problem, for the case when $D^n(y)$ is replaced by $D^n(y) - \sum_{1}^{n-1} D^r(y).K_r$, where $K_r > 0$, and have considered various particular cases. Incidentally we have proved the reality of roots of a class of algebraic equations which does not appear to be capable of being tackled directly.

K. S. Ramamurthi, Tambaram, Madras.

On Positive Valued Functions of a Real Variable.

A function $f(x)$ is said to be concave if $f\{(\sum x_i)/n\} \geqslant \{\sum f(x_i)\}/n$ and convex if the inequality is reversed.

$f^a(x)$ denotes the function got by raising the original function $f(x)$ to the power a.

It can be shown that for positive valued concave functions the values α such that $f^{\alpha}(x)$ is concave is an interval of which the lower end point is zero. The higher end point α is called the *order of the* function.

Similarly for convex functions $F(x)$ there is a lower limit of numbers α for which $F^{\alpha}(x)$ is convex. This will be $\leqslant 1$. This will be called the order of the function.

The problem of extending the notion of order to general positive valued functions is also studied in the paper.

K. S. K. Iyengar, *Bangalore.*

1. *Generalisation of a problem of Erdos and Grunwold;*
2. *A problem in function space;*
3. *The existence of Frullani's Integral.*

C. N. Srinivasiengar, Bangalore.

Non-Differentiable Functions II.

Some further remarks are made on the properties of such functions, with particular reference to Peano's function.

S. Minakshisundaram, Madras.

On the Roots of a Continuous Non-Differentiable Function.

Let $f(x)$ be a continuous non-differentiable function defined in $0 \leqslant x \leqslant 1$, whose least and greatest values are L and U. If $S(\alpha)$ denotes the set of points x for which $f(x)=\alpha$, then for almost all values of α, $L \leqslant \alpha \leqslant U$, $S(\alpha)$ is non-enumerable and of measure zero. More precisely if

A = the set of points α for which $S(\alpha)$ is of positive measure,

B = the set of points α for which $S(\alpha)$ is of zero measure and non-enumerable,

C = the set of points α for which $S(\alpha)$ is at most enumerable, we have then the following

THEOREM :—

A is at most enumerable

$$|B| = U - L$$

$$|C| = 0$$

where $|X|$ = measure of the set X.

For Weierstrass' Non-Differentiable function $W(x) = \sum a^n \cos b^n x$, *$C$ is not empty.* In fact S(L) and S(U) are enumerable. Also the proper maxima and minima of $W(x)$ are the only values α for which $S(\alpha)$ can be enumerable, so that C is at most enumerable for $W(x)$.

P. D. **Shukla**, Lucknow.

On the Derivates of a function of Denjoy

In the present paper the following function defined by Denjoy,

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{(n!)^{n-1}} \cos (n!)^n \pi x, \; 0 \leqslant x \leqslant 1,$$

has been considered. Denjoy used a geometrical method for proving the non-differentiability of the function and obtaining some of its properties. I have used an analytical method to determine the actual values of the derivates at points in (0, 1), and in particular have been able to find out the set of points where the function possesses one-sided infinite differential coefficients.

D. P. **Banerji**, Mymensingh.

On the properties of the functions which are self-reciprocal in Hankel's transform.

The following properties are a selection of results from the paper :—

(i) If $f(x)$ be R_μ then $\dfrac{\sqrt{\pi}\; p^{\mu - \frac{3}{2}}}{2^\mu \Gamma(\mu + \frac{1}{2})} \varphi(p)$ is $R_{\mu-1}$ where $\varphi(x)$ is the operational image of $f(x)\, x^{\mu - \frac{1}{2}}$

(ii) If $f(y)$ be R_μ, the operational image of

$$y^{\mu - \frac{1}{2}} f(y) \text{ is } \frac{\Gamma(\mu + \frac{1}{2})\, 2^\mu}{\sqrt{\mu}}\; p^{-(\mu - \frac{3}{2})}\, g(p) \text{ where } g(p) \text{ is } R_{\mu-1}$$

(iii) If $f(x)$ be R_μ then

$$\varphi(x) = a^\alpha\, g\!\left(x a^{2\alpha}\right) \pm \frac{1}{a^\alpha}\, g\left(\frac{x}{a^{2\alpha}}\right)$$

is $\pm R_\nu$ when $a > 0$, $\alpha > 0$

V. Ganapathy Iyer, Annamalainagar

On Maximal Integral Functions.

A "maximal integral function" $f(z)$ of order ρ and type d ($\rho > 0$, $d > 0$) is defined as a function for which

$$\lim_{|z|\to\infty} \frac{\log |f(z)|}{|z|^\rho} = d$$

as $|z|\to\infty$ outside a system of circles the sum of whose radii is finite. If $n(r, a, \theta)$ be the number of zeros of $f(z)-a=0$ in the sector common to $|z|\leqslant r$ and an angle of magnitude θ, I prove that $\underline{\lim}_{r\to\infty} \frac{n(r, a, \theta)}{r^\rho} \geqslant H(\theta)$ where $H(\theta)$ is a constant depending only on θ.

If, in addition. $f(z)$ has only simple zeros and $[z_n]$ be these zeros, the relation

$$\lim_{n\to\infty} \frac{\log |f'(z_n)|}{|z_n|^\rho} = d, \qquad \dots \quad (1)$$

holds, then I show that any function $g(z)$ of order ρ and type less than d satisfies the interpolation formula

$$\frac{g(z)}{f(z)} = \sum \frac{g(z_n)}{f'(z_n)} \frac{1}{z-z_n} \qquad \dots \quad (2)$$

Conversely, if a function $f(z)$ of ordèr ρ and type d has only simple zeros and the relation (2) holds for all functions $g(z)$ specified above. then $f(z)$ is a maximal integral function for which (1) holds.

S. M. Shah, Aligarh.

A relation between the zeros and the maximum modulus of an Integral Function.

In this note I prove the Theorem :— If $F(z)=e^{g(z)}f(z)$ is an integral function of integral order $p \geqslant 1$ and $f(z)$ is a canonical product of genus p then

$$\underline{\lim}_{r\to\infty} \frac{\log M(r, F)}{n(r)\,\phi(r)} = 0$$

where $$M(r, F) = \operatorname*{Max}_{|z|=r} |F(z)|$$

$n(r, F)=n(r, f)=n(r)$, the number of zeros of $F(z)$ in and on $|z|=r$ and $\phi(x)$ is any positive continuous, non-decreasing function of the real variable x such that

$$\int_\alpha^\infty \frac{dx}{x\,\phi(x)} \text{ is convergent.}$$

R. S. Varma, Lucknow.

1. *Some infinite series involving Sonine's polynomial.*

In this paper I have discussed series of the type

$$\sum_{r=1}^{\infty} b_r \, T_r^n (x).$$

2. *Some infinite Integrals involving Bessel functions.*

In the present paper some infinite integrals involving Bessel functions are first investigated. Some interesting special cases are then discussed.

Hari Shanker, Delhi.

On an integral representation of Weber's parabolic cylinder function and its expansion in an infinite series.

The object of this paper is to obtain an integral representation of $D_n(z)$ in the form

$$e^{-\frac{1}{4}z^2} D_n(z) = -\frac{\Gamma(n-m+1)}{2\pi i} \int_{\infty}^{(0+)} e^{-\frac{1}{4}(z+t)^2} D_m(z+t)(-t)^{m-n-1} dt$$

holding for arbitrary values of m, and to expand $D_n(z)$ in an infinite series of the type

$$e^{-\frac{1}{4}z^2} D_n(z) = \Gamma(m+1)\, \Gamma(n+1) \sum_{r=0}^{\infty} \frac{(-1)^r D_{n-r}(z)\, D_{m-r}(z)}{\Gamma(n-r+1)\, \Gamma(m-r+1)\, r!}$$

S. Minakshisundaram, Madras.

On Partial Differential Equations of the Parabolic Type.

Using Fixpunktsatz of Schander, it can be shown, that we can find a function $u(x, t)$ expressible in the form

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin nx$$

the series on the right converging uniformly and satisfying the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = f(x, t, u)$$

almost everywhere, in $0 \leqslant x \leqslant \pi$ and $0 \leqslant t \leqslant T$, and fulfilling the boundary conditions

$$u(0, t) = u(\pi, t) = 0, \ t \geqslant 0$$
$$u(x, 0) = u_0(x) \equiv 0.$$

provided $f(x, t, u)$ is a bounded measurable function continuous with respect to u and defined in the domain

$$0 \leqslant x \leqslant \pi, \ 0 \leqslant t \leqslant T, \ |u| < M.$$

If $f(x, t, u)$ is a monotonic increasing function of u, there can be only one solution.

K. V. Iyengar, Bangalore.

1. *Simpler proofs and some more theorems connected with Jordan-Brower theorem.*

The object of this note is to extend the method which the author used recently to give a simple proof of the Jordan-curve theorem to analogous theorems in three dimensions. Some additional results are obtained of which the following are cited as examples.

(i) The Smoothness (i. e. glatt-heif) property of one-dimensional cycles at any point of a Jordan-curve in three and higher dimensions.

(ii) If P and Q, two points on different boundary continua of a region say in 3-dimensions are joined by a Jordan arc running inside the region, then the one dimensional Betti-number is increased by unity.

2. *A Note on Poncelet's Problem.*

If $P_1, P_2, \cdots\cdots P_n$, are points on a conic S' such that $P_r P_{r+1}$, $r = 1, 2, \cdots n-1$ are all tangents to another conic S, then it is well known that $P_1 P_n$ touches a conic of the type $S + \Delta_n S' = 0$. The object of this note is to obtain the recurrence formula of Δ_n by simple geometric method without making any use whatever of elliptic functions.

3. *The projective interpretation of the ϕ-conic and corresponding covariants of two quadrics in higher dimensions.*

Various projective interpretations of the same have been given by several authors; but none of them, concern the case when the quadrics are given by their Polarities (according to the Von Staudt definition,) and in a geometry devoid of the continuity axiom (i. e. Pascalian geometry.) In this paper this problem is solved; e.g. the polarity corresponding to the ϕ-conic of two conics S_1 and S_2 is obtained as follows.

Let p be any plane. P_1 and P_2 their poles w. r. t. S_1 and S_2 respectively, P_{12}, and P_{21} *their* polars w. r. t. S_2 and S_1 respectively, and P_{121} and P_{212}, the poles of *these* w. r. t. S_1 and S_2 respectively. Then the polar of p, w. r. t. the ϕ-conic is the point of intersection of P_1P_{121} with P_2P_{212}.

M. N. Narasimha Iyengar, Mysore.

1. *On a problem connected with tetrahedra of given faces.*

2. *On conics having a common self-conjugate triangle.*

3. *On distances between special points associated with a triangle and a tetrahedron and their applications to the geometry of conics and conicoids.*

Karam Chand

Applications of Desargue's Theorem

V. Rangachariar, Patna.

On the nature of the conicoids through a quartic curve and touching a given plane.

Three conicoids can, in general, be drawn to pass through a given quartic curve and to touch a given plane. The object of this paper is to discuss the nature of the three conicoids corresponding to a given plane. There are three paraboloids of the system two of which are elliptic and the third hyperbolic. Of the three conicoids touching a given plane one is always real and is a hyperboloid of one sheet. The nature of the other two conicoids depends on the position of the plane relative to the two elliptic paraboloids.

Ram Behari, Delhi.

Rectilinear Congruences whose foci are at infinity.

In this paper necessary and sufficient conditions that a rectilinear congruence may have its foci always at infinity are obtained. Some instances of congruences which have both foci at infinity have also been considered.

B. Ramamurti, Ajmere.

On a certain correspondence between lines in [4] and planes in [6].

In a paper, published recently by the author (A generalisation of the null pencil *Jour. London Math. Soc. Vol. 13, 1938, 162-167*) a correspondence is established between points in $[n+1]$ and certain special $[n-1]$s of $[2n]$. The object of this paper is to study in detail the specialities which follow in the case $n=3$.

A. Narasinga Rao, Annamalainagar.

On the Topology of line-elements in the inversive plane.

The connection between the group of a geometry (in the sense of Klein's *Erlanger Program*) and the topology of the operand-manifold arises from the need for a suitable closure by adjunction of ideal elements so as to secure that each transformation of the geometry shall effect a (1, 1) correspondence of the elements with themselves. It is shown that the oriented line elements of the inversive plane are topologically equivalent to the points of a projective 3-space while the non-oriented line elements are homæomorphic with a lenticular 3-space whose Poincare Group is cyclic and of order 4. Both these are of the type known as fibred spaces, the fibres corresponding to "stars".

P. Samuels Lall, Lahore.

On an extension of the method of least squares.

(i) The principle of the method of Least Squares is stated.

(ii) In certain cases it may happen that one of the quantities x_n to be determined, though unknown, is suspected of lying between certain limits. It then becomes necessary to determine the remaining unknown quantities in terms of x_n for a set of possible values lying within an interval and further considerations lead to the most plausible set of values of the unknowns.

(iii) A simple means for an extension of the Least square method is given for such cases. An example is given in which the above method is applied.

A. A. Krishnaswami Ayyangar, Mysore.

1. *A synoptic view of some small sample distributions.*

This paper expounds a derivation of the well-known distributions of the mean, the standard deviation, correlation-coefficient and a few other statistical derivatives of small samples by a uniform procedure based on linear transformations and Dirichlet integrals without appeal to n-dimensional geometry.

2. *Some problems on mathematical expectation.*

It was sometime ago noticed in our journal that the mathematical expectation of a drawing from a bag containing coins is

independent of the manner of drawing. We show here that this statement is not true in general and the apparent cases of success are due to fortuitous algebraic circumstances.

M. V. Jambunathan, Mysore.

The Moments of a Frequency Distribution employing 'Repeated Ogives'.

Let f_r denote the frequency of the occurrence of the value $r\,(r=1, 2, 3, \ldots\ n)$ and let the "backward ogives" or frequencies summed cumulatively from the end f_n be denoted by ${}_1G_r$, i.e. let $\sum_r^n f_m = {}_1G_r$. Similarly, let

$$\sum_r^n {}_1G_m = {}_2G_r;\quad \sum_r^n {}_2G_m = {}_3G_r;\quad \sum_r^n {}_3G_m = {}_4G_r,\ \ldots\ \ldots\ \text{etc.}$$

In this paper it is shown that the several *moments* of the frequency distribution are given by the relations

$$\begin{aligned} &M_0 = {}_1G_1;\quad M_1 = {}_2G_1;\\ &M_2 = {}_3G_1 + {}_3G_2;\\ &M_3 = {}_4G_1 + 4\cdot{}_4G_2 + {}_4G_3;\\ &M_4 = {}_5G_1 + 11\cdot{}_5G_2 + 11\cdot{}_5G_3 + {}_5G_4;\ \text{etc.} \end{aligned}$$

It will be noticed that the results obtained by the above method are symmetrical, a feature which the 'summation method' of computing the moments is lacking in.

M. R. Doresamiengar, Mysore.

Variation of Pareto's Law—a sequel.

"Pareto is a great name and analytical laws in Economics are few".........naturally, a Variant of Pareto's statement that is proposed, has to be expectant of criticism. The present paper is a continuation of the author's original paper, "Variation of Pareto's law" and seeks to explain the *a priori* reasoning behind the statement, $\frac{dN}{N}\Big/\frac{dx}{x} = -kx$ and also sets out the mathematical consequences of other variants, like $\frac{dN}{N}\Big/\frac{dx}{x} = -kx^2$; $\frac{dx}{x}\Big/\frac{dN}{N} = -kN$.

M. V. Vaidyanatha Sastri, Hyderabad.

Heliacal rising of the moon

The moon's visibility for the first time in the evening after new moon determines the beginning of the Hijri month. This is of special importance in preparing the official Hijri calendar. The ancient Hindu methods based on the elongation of the moon, of determining the day on which the phenomenon occurs are discussed in this paper. It is shown that the altitude criterion is the safest one for deciding about the visibility of the moon and that her visibility can be taken as fairly certain when her altitude is greater than eleven degrees.

H. Subramani Iyer

Precession in Hindu Astronomy

B. S. Madhava Rao, Bangalore.

On the reduction of dynamical equations to the Lagrangian form

It is shown that in the case where the kinetic energy is quadratic in the velocities the condition for the reduction to Lagrangian form is deduced very simply by using tensor methods. Other cases are also considered, and the application of the method of the last multiplier to the problem is investigated.

2. *On an invariant relation of dynamical systems*:

A. Wintner (*Q. J. Math.* Vol. 7, (1936), p. 214) has derived a new elementary invariant relation valid along those solutions of certain dynamical systems which run in the sub-space $h=0$ of the phase-space. The full significance of this in the general case from a geometrical point of view is discussed in this paper.

K. Nagabhushanam, Waltair.

On a Property of the Lagrangian.

In an affine manifold of states and time, following the invariant definition of the Lagrangian given by me, I here show that for the singular curves of the curl of (X_i),

$$\iint \cdots \int L\, \alpha^{2n+1}\, d\omega$$

is an integral invariant for all co-ordinates with $t = x^{2n+1}$, and that when

$$x^r = q^r, \quad x^{r+n} = p_r, \quad x^{2n+1} = t,$$

this takes the form

$$\iint \cdots \int \left(p_r \frac{\partial H}{\partial pr} - H\right) dq^1\, dq^2 \cdots dq^n\, dp_1\, dp_2\, dp_n\, dt.$$

B. R. Seth, Delhi.

Strain in a spherical shell turned inside out.

The problem of a spherical India-rubber shell turned inside out is one which cannot be solved by the ordinary Mathematical theory of Elasticity which deals with only small strains. In the present paper the theory of Finite strain which has been developed by me in some recent papers* is applied to this problem, and the following results have been obtained:

(i) The increase in the external radius of the shell is about 2 per cent.

(ii) The thickness decreases by about 0·07 per cent.

(iii) The radial stress $\widehat{rr}$ does not vary very much with r.

(iv) The absolute value of the cross-radial stress $\widehat{\theta\theta}$ is greatest at the inner boundary.

B. Ramamurti, Ajmer.

A theorem on Eddington's E numbers, and its relation to Dirac's theory of the electron.

Four numbers E_1 E_2 E_3 E_4 so that $E_r^2 = -1$ and $E_r E_s = -E_s E_r$ $(r \neq s)$ generate 16 linearly independent numbers including -1. Any number A formed from a linear combination of them is said to be singular if another number B could be found so that $AB=0$. The totality of E numbers, which form a linear ∞^{16} system, and hence may be said to be of rank 16 is reduced to a lower rank by multiplication by a singular member. The object of this note is to prove that the minimum rank to which the system of E numbers could be reduced is 4.

The E numbers are equivalent to the 16 numbers generated by the four symbols introduced by Dirac, in his relative equation of the electron. Sauter† has solved Dirac's equation without any matrix representation of Dirac's symbols, by taking any function of the 16 symbols and reducing it to rank 4 by multiplying by a properly

* *Phil. Trans. Roy. Soc.* A, 234 (1935), 231-64; *Proc. Roy. Soc.* A, 156 (1936) 171-92 *Phil. Mag.*, Ser. 7, 27 (1939), 286-93.

† Sauter: *Zeitsohrift fur Physik*, vol. 63, pp. 803 (1930).

chosen symbol. Our result shows that it cannot be reduced further. Probably this explains why Dirac's state function necessarily involves four components.

Gunjikar and **Ozu**, Bombay

Joules Law of minimum dissipation.

The paper shows the invalidity of the conclusions usually drawn from the minimum property of the expression $\sum C^2R$ for a network of steady currents.

A. H. Querishi, Peshawar.

The case for and against mathematics.

A. N. Singh, Lucknow.

A Sanskrit Translation of Euclid.

A Sanskrit Translation of Euclid was made in 1718 A.D. by Samrat Jagannatha, the court-astronomer of Maharaja Jai Singh of Jaipur. This translation has been published by K. P. Trivedi. The translation is based on an Arabic version. Trivedi thought that the Sanskrit translation was made from the Arabic version of Nasiruddin, but a closer examination of the work reveals that the Sanskrit translation is based on the Arabic version of Ishaq. The present paper gives a short account of the life of the translator, Samrat Jagannath, and of some interesting points connected with his translation.

V. P. Venkatachary, Hyderabad.

India's Contribution to continued fractions.

An attempt is made in this paper to show how the ancient Hindu Mathematicians originated the theory of Continued Fractions. Arya Bhatta (born 476 A.D.) was the inventor of this theory. His method of successive reductions corresponds to the method of Continued Fractions. Later mathematicians followed him with improvements and extensions. Bhaskara's method of solving linear indeterminate equation is more akin to the modern method. Brahmagupta originated the theory of recurring continued fractions by the enunciation of his remarkable principle known as 'Principle of

Composition of roots.' He could solve $x^2 - Ny^2 = 1$ (N being a non-square integer) when he knew the solutions of $x^2 - Ny^2 = k(k = \pm 2, \pm 4, -1)$.

Bhaskara tried to solve $x^2 - Ny^2 = 1$ when the solutions of $x^2 - Ny^2 = k$ (any integer) were known to him. In this connection he stated his famous method 'Chakravala' which embodies the whole theory of recurring continued fraction. The important properties of recurring C.F.'s are deduced from 'Chakravala' or the 'cyclic method'. It is shown in the end that Bhaskara's method of solution is very simple and claims superiority to any other method, ancient or modern, in the matter of practical computation. A few examples are worked out to elucidate the various points.

GLEANINGS

THE CIRCLE

By Christopher Morley

Few things are perfect : we bear Eden's scar ;
Yet faulty man was godlike in design
That day when first, with stick and length of twine,
He drew me on the sand. Then what could mar
His joy in that obedient mystic line?
Approximating with a zeal divine
He called π 3-point-14159
And knew my lovely circuit $2\pi r$!

A circle is a happy thing to be—
Think how the joyful perpendicular
Erected at the kiss of tangency
Must meet my central point, my avatar.
And lovely as I am, yet only 3
Points are needed to determine me.

[From "*Scripta Mathematica*" October 1938.]

Symposium on Generalized Geometry, including Relativity and Field Theories

Chairman: PROF. D. D. KOSAMBI, *Fergusson College, Poona*

Initiating the Symposium, the Chairman said:—

GENTLEMEN!

This symposium is intended to be a set of twenty-minute lectures by six people on their own or connected work in branches of modern geometry, mainly the kind of geometry associated with tensor analysis. We are not in the fortunate position of Dr. Pillai, whose symposium deals with a single clearly defined and well delimited topic—Waring's problem; a problem which is almost finished because of his own efforts though regarded as almost insoluble ten years ago; and yet a problem which has enough interest left to make it ideally suitable for the purpose of a symposium. Today's lectures will be on tensor analysis, differential invariants, group theory, and topology.

You will then wonder whether it was worth while to tie together so many scattered branches of a vast subject. As a matter of fact, there is a profound fundamental unity in all that will be said today, and the lifework of one very distinguished mathematician has gone to show this unity.

I refer to Prof. Elie Joseph Cartan of Paris, who completed his seventieth year in April, and to whom I propose—with your permission—to dedicate this symposium in appreciation of his magnificent contributions to science. I trust that you will agree with me in the hope that the tribulations of the war in which his country is engaged will not prevent his putting the finishing touches to a pioneer work which constitutes perhaps the most important individual performance in differential geometry in our time.

Let me add that work in geometry in India is not confined to the topics we have chosen. You must all have heard of the work in algebraic geometry performed by Dr. Vaidyanathaswamy and his pupils; also, the work on topological differential geometry, after Blaschke, so brilliantly carried out by Prof. Shyamadas Mukhopadhyaya and his co-workers. Lastly, the relativitists and field

theorists deserve, and originally had, a symposium to themselves. This was cancelled because of the crowded program, but you will hear two of the best today, and Dr. N. R. Sen has inaugurated the conference, so that we have not lost much after all.

The Chairman then moved and conference passed a unanimous resolution of sympathy for Prof. Tullio Levi-Civitá of Rome, who, at the end of a long and distinguished mathematical career, is now suffering from the newly developed anti-semitic policy of the Italian Government. The Symposium itself was dedicated to Prof. Elie Joseph Cartan of Paris in honour of his seventieth birthday.

Prof. KOSAMBI then proceeded to show how simple the elementary notions of tensor analysis can be when taken without any of the formal apparatus of algebraic invariant theory. Starting with the transformation laws for covariant [force-like] and contravariant [velocity-like] vectors, we can develop a tensorial differentiation and calculate all the differential invariants associated with a system of differential equations of the second order. As a special case, one can consider the systems involved in the theory of relativity. These can be generalised in one direction by considering all the systems that admit the Lorentz group of transformations. It was shown that systems of second order differential Equations of this latter type not only include all equations proposed by Milne, but also make clear what the utmost possibility is as regards an "intrinsically expanding" universe. [the technical portions of the lecture will be published in a forthcoming issue of the *Journal of the Lond. Math. Soc.*]

DR. B. S. MADHAVA RAO (Bangalore) read an elaborate paper † which dealt with the intimate connection existing between *Generalised geometry and physical theories*. The old idea that physical space is identical with three-dimensional Euclidean space has given place, with the advent of Relativity, to the notion that geometry, in so far as it introduces the determination of measure in the space-time world, has essentially an empirical character. Thus the notions of Riemannian Geometry have become of fundamental importance in physical attempts at geometrisation of natural phenomena. With the appearance of Quantum Mechanics the geometry of abstract spaces, specially the Hilbert space, has come to have various applications in physics. In some recent theories devised to get over the fundamental

† This will appear in the next issue of the STUDENT.

difficulties of quantum mechanics, one can see a trend towards ideas of non-Riemannian geometry and topology.

Even the old classical mechanics has been influenced by the newer geometrical notions. Thus the notions of configuration space and phase space have led to the theorems of classical mechanics as geometrical theorems in Riemannian space. The properties of dynamical systems "in Grossen" have led to topological characterisations of dynamical motions.

The paper dealt with several such theories drawn from Relativity, Cosmology electro-magnetic field theory, Quantum Mechanics and Nuclear Physics.

FATHER C. RACINE (Madras) then spoke on *Group Theory and Generalized Geometries.* A summary of his address is given below:—

Prof. E. Cartan has shown how it is possible to develop *any* differential geometry upon the basis of the theory of finite transformation groups. This basis is by far the most natural and the broadest possible. It has proved also to be the best guide in investigations about the possible generalizations of Geometry and the *complete* set of differential invariants, if not their analytical expressions, in each case. This lecture aims at outlining the ideas and methods of Prof. E. Cartan.

In the first part, the notion of fundamental space and of the moving frame of reference—" Repère Mobile"—is dealt with, along with a method to find out the differential invariants of any variety embedded in such a space.

In the second part the notion of "differential connectivity" is defined, and a method for generalizing any fundamental space developed. Exactly as a sphere can be considered as the limit of a polyhedron, so any generalized space can be considered as the limit, under certain conditions, of a certain assemblage of small portions of a fundamental space. A few examples such as spaces with euclidean, projective, conformal, ... connectivity are discussed. It is shown how all their differential invariants may be deduced from the equations defining the differential connectivity.

A complete bibliography has been published in the Jubilee Volume "Selecta" (Gauthier-Villars 1938). A partial one, but quite sufficient for a first contact with these fields of Research, is to be found in Cartan's Volume " La théorie des groupes finis et continus

et la Géométrie différentielle traitées par la méthode du Repére Mobile" (Paris, Gauthier-Villars, 1937). Reference may be made also to the fascicle 42 of "Mémorial des Sciences Mathématiques". It deals with the theory of finite and continuous groups from a new point of view but may be taken as a very good introduction to the subject.

Prof. N. G. SHABDE (Nagpur), spoke on *Projective Relativity*. The problem of unified field theory is to find a geometry, representing both gravitational and electromagnetic phenomena. A generalized theory making use of five homogeneous co-ordinates is given by Schouten and van Dantzig. Introducing analogues of tensors, called projectors, the projective curvature tensor $N_{\dot{p}\dot{q}\dot{r}\dot{s}}$ is constructed. From this the scalar N is obtained by methods similar to those in ordinary tensor-calculus. The variational principle $\delta I = 0$ where

$$I = \int\!\!\int\!\!\int\!\!\int\!\!\int N\sqrt{G}.\ dx^0\, dx^1\, dx^2\, dx^3\, dx^4$$

gives the gravitational and electromagnetic field equations. As has been shown by the lectuler (*Phil. Mag.* Vol. XXII), an application of a variational theorem by Emmy Noether to I gives the indentities between the field equations in the field theory of Schouten and van Dantzig.

Prof. V. SEETHARAMAN (Annamalainagar) who spoke next outlined briefly two *Methods of generating differential invariants*, the one due to Prof. D. D. Kosambi and the other to Prof. Elie Cartan of Paris. For illustration he considered a path-Space of order two, viz. one in which the paths are defined by differential equations of order three,

$$x'^{3i} + \alpha^i\,(x,\ x',\ x'',\ t) = 0.$$

The first method referred to, consists in first deriving a basis for all invariantive operators of the space, and then by alternating these taken two at a time, all the invariants are obtained. That the invariants so obtained furnish the basis for all the invariants of the space can be seen even by a direct appeal to the equations of the paths and their equations of variation. The invariants should be the derivators of α^i of various orders. The first derivatives are in essence the primary tensors appearing in the equations of variation and further derivatives lead to $\gamma^i_{j(2)k}$ and $\underset{1}{\nabla}_k \underset{1}{\nabla}_j \gamma^i_r$ as the only invariants of the second and third rank respectively.

Cartan's method consists in calculating the Poisson parentheses of certain Pfaffians, that are defined in terms of dx^i, $d\dot{x}^i$ and $d\ddot{x}^i$ and another which gives us the connection.

He concluded by pointing out that both these methods gave us the same set of basic invariants, a result naturally to be expected since both the methods reduced to alternations of fundamental operations.

Prof. K. VENKATACHALIENGAR (Bangalore) concluded the Symposium with a talk on *Point Set Topology* or *Continuity Geometry*, a subject whose foundations had been laid by Cantor but which had developed remarkably in recent times. One problem is to define space in a topologically invariant manner with a minimum set of axioms. From generalised topological space we derive the Hausdorff-Kuratowski, Frechet. Hausdorff, regular, normal and metrical spaces. The last among these has been topologically characterised by the property that "A regular space (*i.e.* one in which every neighbourhood of any point P contains a closure of another neighbourhood of P) with an enumerable basis is metrisable".

As another example may be given Lebesgue's definition of the "dimension" of a set: a set A is n-dimensional if every covering of A by means of closed sets (of arbitrary fineness) contains at least one point common to $(n+1)$ of them.

The characterisation of a Jordan curve in an R^2 by means of its outer properties is as follows: It must divide R^2 into exactly two regions and all points of the set must be reachable boundary points for both the regions. Brouer has shown by means of examples that if the points of the set are not reachable, the points need not form a Jordan curve.

Symposium on Waring's Problem

Chairman: PROF. S. S. PILLAI, (*Annamalainagar.*)

Giving a historical survey of Waring's Problem, Dr. Pillai said :—

The Theory of Numbers is the only branch of mathematics in which many of the problems solved and unsolved may be explained to laymen, and even broadcasted in newspapers. It is no surprise that Vinogradow's sensational result, that every large odd number is the sum of three primes, got a place in the *Sunday Times*. In no other branch of Mathematics, can one find problems which are more than

two-thousand years old and yet defy all attempts to solve them. Many of the majestic peaks of Number Theory are such that any one can have a glimpse at them, but even the greatest masters of the science have no hope of reaching them at present.

Two of such outstanding peaks are 'Waring's problem' and 'Goldbach's theorem', named after the men who had the first glimpses of them. Many are the attempts made to climb up to these. It is really astonishing that a single man's partial success has eclipsed the results obtained by all others in both these problems. Today, Vinogradow has the glory of reaching heights which are marvellous compared to those conquered by all others. Why, in Goldbach's theorem he has reached a peak which is as important as the main one! Though he has not reached the final goal in Waring's problem, the height that he has reached is amazing, and this has enabled other investigators to reach a subsidiary peak.

In the year 1640, Fermat stated, among other results, that every number is the sum of four squares. Though his statement implied that he had a proof by his famous method of infinite descent, it was Lagrange who published a proof for it in 1770. Suppose we want to express n as the sum of squares, say, in the form

$$n = x_1^2 + \cdot \cdot \cdot + x_s^2.$$

To start with, we have no reason to expect that the minimum value of s will not increase with n. Fermat's theorem means that it is possible to find s independent of n, and that its value is 4. This is certainly one of the most beautiful results in Arithmetic.

Waring's Problem is the generalisation of this result for k-th powers. In the year 1770, Waring, Lucasian Professor of Mathematics in the Cambridge University, stated, in his 'Meditations Algebraic,' that every number is the sum of 4 squares, 9 cubes, 19 biquadrats and so on. Let $g\ (k)$ be the least value of s such that every number is the sum of s k-th powers; *i.e.* the equation

$$n = x_1^k + x_2^k + \cdot \cdot \cdot + x_s^k$$

is solvable for every n. Here also, in the first instance, we do not see whether the least value of s will increase with n or not. Waring's assertion means not only that $g\ (k)$ is independent of n, but also that $g\ (2) = 4$, $g\ (3) = 9$, $g(4) = 19$, and so on. But Waring's result was only a conjecture, and he had no proof for it.

In 1772, Euler, the son of the famous Leonhard Euler, showed that $l \cdot 2^k - 1$ requires $2^k + l - 2$, kth powers, where $l = [(3/2)^k]$, and thereby

proved that $g(k) \geq 2^k + l - 2$. The next advance was made in 1858 when Liouville proved that $g(4) \leqslant 53$. In the years up to 1909 upper bounds for $g(k)$ were evaluated and improved for all k up to 10 excepting 9.

The year 1909 is an important one in the history of the problem. In that year, Wieferich proved that $g(3) = 9$. Landau proved that every large number is the sum of eight cubes, and above all Hilbert proved the existence of $g(k)$. Speaking about the problem and its solution by Hilbert, Professor Hardy says "Within the limits which it has set for itself, it is absolutely and triumphantly successful, and it stands with the work of Hadamard and de la Vallee-Poussin, in the theory of primes, as one of the landmarks in the modern history of the theory of numbers."

Landau's result is important in that it was responsible for the introduction of another and more fundamental function $G(k)$ by Hardy and Littlewood. Landau's 'singularly beautiful theorem' means that the number of numbers which require nine cubes is finite. As a matter of fact, recently Dickson has proved that 23 and 239 are the only numbers which require as many as 9 cubes. $G(k)$ is defined as the least value of s such that every sufficiently large number (every number from a certain integer onwards) is the sum of s, k-th powers. Since numbers of the form $8m+7$ cannot be expressed as the sum of three squares, Fermat's result means that $g(2) = G(2) = 4$. Landau's result takes the form $G(3) \leqslant 8$.

Hilbert's proof was a pure existence proof. The next advance was due to Hardy and Littlewood. In the years from 1919 to 1927, they attacked the problem in a series of memoirs. They discovered a method of applying Cauchy's theorem to the problem. It may be remarked that the 'Farey-dissection' of the circle, which plays a fundamental role in all subsequent investigations, first appears in Hardy-Ramanujan's memoir on Partitions. By their important method, besides finding upper bounds for $G(k)$, Hardy and Littlewood were able to find asymptotic formula for the number of representations of a number as the sum of s k-th powers, when s is large. In these memoirs, they introduced many fundamental ideas and devices, without which subsequent advances would have been impossible. The final result about $G(k)$ which they proved is that

$$\overline{\lim}\ G(k) / (k \cdot 2^k) \leqslant \tfrac{1}{4}.$$

Though the success of their method was phenomenal in dealing with $G(k)$, they did not apply it to obtain an upper bound for $g(k)$.

But Vinogradow found out a new method to prove Hilbert's theorem, and the consequences of that method are given in Landau's *Vorlesungen über Zahlentheorie.* The result proved there, is that

$$\overline{\lim}\ g(k)/(k\cdot 2^k) \leqslant \tfrac{1}{2}.$$

In 1933, R. D. James applied the Hardy-Littlewood's method and reduced the constant in the above limit to 1/4 and improved upon the upper bounds for $g(k)$, given by earlier investigators for small values of k.

In the year 1935, Vinogradow was able to prove the wonderful result, that

$$\lim G(k)/(k \log k) < 6.$$

Though this is a tremendous improvement upon Hardy and Littlewood's result, it is far from the final solution of $G(k)$. Yet it prepared the ground for the solution of $g(k)$, which is final in a sense.

By evaluating the constant in Vinogradow's paper, I was able to prove, in September 1935, that

$$g(k) = 2^k + (3/2)^k + O(4/3)^k.$$

The result was communicated to the conference of the Indian Mathematical Society held at Delhi in December 1935. But in the third week of December 1935, I was able to prove that

$$\text{if } \{(3/2)^k\} < 1-(l+3)/2^k, \text{ then } g(k) = 2^k + l - 2, \text{ when } k > k_0.$$

The result was published in the March issue of *Journal of the Annamalai University*, and advanced reprints were sent to some mathematicians including L. E Dickson about the 10th February 1936 (perhaps, before the 10th).

By refining the above method, I brought down k_0 to 7 in the last week of January 1936, and published the proof when $k_0 > 8$ in the Journal of the Indian Mathematical Society. The fact that this problem should depend upon the fractional part of $(3/2)^k$ was a great surprise to me.

The proof of the above result when $k_0 > 7$ together with an exact formula for $g(k)$ when $\{(3/2)^k\} > 1-(l-1)/2^l$ was published by L. E. Dickson in the July (1936) issue of the *Journal of the American Mathematical Society*. The latter result was discovered independently by me.

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* It may be mentioned that the result was proved without applying Dickson's method of ascent.

These results form an 'almost complete solution' of the problem of $g\ (k)$ in a sense. Though there is no doubt that $\{(\tfrac{3}{2})^k\} \leqslant 1-(l+3)2^k$ for all $k \geqslant 4$, unless it is settled one way or the other, we cannot say that the problem of $g(k)$ is completely solved.

SAMBASIVA RAO (Waltair) said that Vinogradow's success in Waring's Problem was partly due to successfully using the following function: Let $H_{s,k}\ (N)$ denote the number of integers $\leqslant N$ that are expressible as the sum of s k-th powers. By considering integers representable as the sum of s k-th powers in a particular fashion, he proved that $H_{s,k}\ (N) > C.\ N^{1-\frac{1}{6k^2}}$ for $s > c_1 k \log k$. Any improvement of this inequality leads to an improvement in Vinogradow's result. Attempts are already being made by Davenport and others in this direction. Davenport's results lead to startling improvements in Waring's Problem in the case $k=3$ and 4. Continuing Davenport's work, the speaker was able to improve previous results in the cases $k=5,\ 6,\ 7$.

The solution of Goldbach's problem for odd integers by Vinogradow opened up a new field. It was immediately proved that every sufficiently large integer is expressible as the sum of $s\ \{=6\ (k \log k)\}$ k-th powers of primes. Following Davenport's lead improvements could be effected in this field also for small values of k.

Another direction in which Waring's Problem was generalised was the representation of integers as sums of polynomial summands instead of as sums of k-th powers, where the variable takes integer values, which may be required to be all primes. Work has been done in this direction by Hua, Pillai, the speaker and others.

Dr. T. VIJAYARAGHAVAN (Dacca) spoke mainly about the work of Dr. Chowla. The following were the main results to which he referred: (1) The existence of numbers which can be expressed as the sum k, k-th powers in any number of ways. (2) The expression of a number as the sum of almost equal k-th powers. (3) If K (x) is the number of numbers $k \leqslant x$ for which $g\ (k)=2^k+l-2$, then $\lim\ K(x) \to 1$ and (4) Mr. I. Chowla's result on the addition of residue classes.

Discussion on the Teaching of Mathematics

Chairman: QAZI MOHAMMAD HUSAIN Esq. HYDERABAD

Prof. J. MACLEAN (Bombay) in initiating the discussion stressed the importance of certain topics which should receive special attention in a first year college course. These were, in his opinion, the function concept and the

functional scale, generalised coordinates, a fuller treatment of curves, the calculus, an idea of things in three dimensions, nomograms and statistics. As things stood at present, the same subjects as in the school course e.g. Algebra, Geometry were continued in much the same way, and little was done to give a new and wider outlook such as was required in the modern world. The criterion which ought to decide the inclusion of a particular topic should be whether it was far reaching, serviceable and unifying. The students should become familiar with the use of the slide rule and learn both the power of the graphical method and its limitations.

Prof. KISEN CHAND (Hyderabad) who followed was against the existence of alternative courses in Intermediate Mathematics, and said that there should be greater uniformity among the different Universities so as to make migration easier. He was for speeding up the existing courses so that by the end of the matriculation stage the whole of Euclid (11 books), progressions and Quadratics would have been finished. The two years of the Inter course provided about 300 teaching hours in which most of algebra, Trigonometry, Calculus, coordinate geometry, mechanics and hydrostatics could be covered. He then outlined the corresponding continuation courses in the degree and post graduate classes.

Prof. MOHD KHAJA MOHIUDDIN (Hyderabad) deplored the wastage of time and effort in the high school stage and said that topics were sometimes taught in the wrong order, the difficult ones first and the simpler later. He cited by way of example the following problem "If $x + \frac{1}{x} - 1$ what is $x^2 + \frac{1}{x^2}$?" which, to the boy who knows nothing of imaginary numbers, started with an impossible hypothesis. He was for a rearrangement of the syllabus on psychological lines.

Prof. B. RAMAMURTI (Ajmere) pointed out that in nearly half the Universities in India calculus was taught in the Intermediate stage and that there were wide differences in standard even among these. In the matter of actual teaching he was for taking full advantage of the students' intuitive ideas and would begin a study of functional relationships and rates of growth on a foundation of graphical work. No loose statements should be made which the student would have later to unlearn. In other words, the method should be rigorous without an exhibition of rigour—rigorous in the long run. In the M. Sc. courses he was for stressing the Group concept and for the inclusion of topics like spherical harmonics.

Prof. C. R. CHATURVEDI (Agra) spoke on the disabilities of Mathematics students at the competitive examinations, a striking evidence of which was that such candidates very often offered a different subject at the competitive examinations. He made out a strong case for candidates offering mathematics at the Indian Accounts and Audit Service Examination being permitted to offer also Elementary Mathematics pointing out that a similar situations holds regarding other subjects and that the marks obtained in Elementary Mathematics by those offering subjects other than Mathematics were already so high that mathematics students could not do much better. He drew attention to the hiatus between the courses of study for the University examinations and competitive examinations. He was also for mathematics being placed on a par with History and other subjects in the matter of marks and for the general knowledge test being placed on the same conditions as the London I. C. S.

After a further discussion it was resolved that " The Managing Committee of the Indian Mathematical Society be requested to urge on the Federal Public Service Commission the need for investigating the various handicaps under which candidates offering mathematics in the Higher Competitive Examinations labour under existing regulations."

Dr. R. VAIDYANATHASWAMI (Madras) said that what was taught in Universities was " word knowledge." It would become Real knowledge only when it was seen in its proper setting with its experience in the world of life. Prof. Maclean's idea was that there should be courses which should catch such deep ideas as the mathematician has to offer for the enrichment of life without going too deep into life—courses fitting a man for a general cultural life. Naturally such a course would deal with topics connected with generalised coordinates, real number, function concept, rates of growth and statistics.

Prof. GUNJIKAR (Bombay) did not like uniformity and felt that each University should make its own experiments so as to allow for the diversity of tastes and interests. He discussed also the purposes served by examinations and how questions should be set so as to serve these purposes.

The Chairman Prof. QAZI MOHAMMAD HUSAIN (Hyderabad) in winding up the debate pointed out the importance of admitting students who had an aptitude for the subject. He described the outlines of a syllabus for the various grades of study in a University.

Minutes of the Business Meeting

held at 12-30 P.M. on 23—12—39.

Chairman :—DR. N. R. SEN, *Ghosh Professor of Applied Mathematics, Calcutta University.*

1. The financial position of the Society was considered and attention was drawn to the need for enlisting more members and subscribers.

Regarding the rates of subscription for members the general body suggested to the Managing Committee an examination of the life composition fees and the consideration, in particular, of the proposal for a uniform rate of Rs. 10/- per annum for all members and a life composition fee of Rs. 150/-

2. A committee consisting of Dr. N. R. Sen, Dr. Vaidyanathaswami, Dr. A. N. Singh with such others as they may co-opt was appointed to consider means for securing a closer co-operation between the Benares, Calcutta and the Indian Mathematical Societies.

3. The question of holding the conferences of the Indian Mathematical Society along with the sessions of the Indian Science congress was considered, but the suggestion received little support from the members of the Society.

Public Lecture on "Meteoric Astronomy"

BY

MOHD. ABDUL RAHMAN KHAN ESQ., F.R.A.S., *Hyderabad*

Meteoric Astronomy, as the name implies, is that branch of Astronomy which deals with the study of meteors. It endeavours to account for their light effects, their nature and origin in space. It involves a study of meteorites as well, which are closely allied to them, being fragments of matter dropped from larger and much more imposing meteors, that have penetrated through the entire depth of our atmosphere and survived its devastating effects. It is thus evident that meteoric astronomy has to seek aid, more or less, from a number of sciences other than astronomy, chiefly, physics, chemistry and geology.

Meteoric Showers

Though meteors have been observed from time immemorial and commented upon by scientists and soothsayers alike, their systematic study commenced only recently and has been occasioned by the interest created in them after the spectacular showers of 1799 and 1833 A.D. In both these showers the radiant, or starting point common to all the meteors of the class, was observed to be close to γ Leonis, hence their designation, Leonids. The former of the above two showers, which occurred on November 12 was carefully recorded by Humboldt while travelling in Cumana in South America; and the latter, a display of surpassing brilliancy, on November 13, by a number of observers in Europe and America. Denison Olmsted and Palmer of New Haven, Con. made a special study of it. According to Olmsted's estimate, no fewer than 240,000 meteors appeared in 9 hours; and Palmer connected the shower with that of 1799, as a recurring phenomenon with a period of roughly 33 years between two consecutive maximum displays.

In this way the Leonid shower was expected to repeat in 1866 and was hailed with great enthusiasm all over the world, when the Leonids began to appear in greater and greater numbers every year from 1864 to 1866. W. F. Denning described the display of November 14, 1866 as the finest observed in England, with 100 meteors per minute. The shower was repeated on the same date in 1867 and subsequent years, but gradually fell off by 1869. After this, H. A. Newton and other investigators ransacked old historical records and prepared a list of conspicuous Leonid showers of past epochs. Newton's list begins with the shower of October 13, 902, long remembered in the Arabic world as "the year of stars", from the fact that the Aghlabid Sultan, Ibrahim ben Ahmed died on the night of the shower, when the Saracen army had just captured Taoramina in Sicily.

[Calculation shows that when the Leonids passed in the neighbourhood of Uranus in 126 A. D., in pursuit of their former parabolic orbit, their path was deflected by the perturbation of that planet into an elliptic one, and their direction of motion round the Sun was completely reversed; so they nowadays approach the Sun to meet the Earth, with an average velocity of 80 kms. per sec. and a period of 33·25 years. The shape and dimensions of this orbit have remained practically intact since that event, but it has undergone a slight

shifting of position in its own plane, and the date of maximum display of the meteors is delayed by one day every 70 years.]

It is a pity that the next expected Leonid shower at the beginning of the present century turned out to be disappointing—as also that of the last (1933) epoch. But interest in Meteoric astronomy is now too well established to be affected by the waywardness of sensational showers Thanks to the labours of Charles P. Olivier (Director, Flower Observatory, University of Pennsylvania) and the team work of the A. M. S. (an institution of international character) visual observations are conducted throughout the year over a vast range of the globe, and are published in the Meteors section of *Popular Astronomy*, Northfield, Minn. The enthusiasm and resources of the Harvard College Observatory under its present Director, Prof. Harlow Shapley and his staff, have enriched meteoric astronomy with methods of the finest accuracy of research, through photography and spectroscopy, and have led to valuable discoveries about the role of meteors in interstellar space, and the density and temperature of the upper regions of our atmosphere inaccessible to balloons manned or unmanned. As a result of these activities, it would be no exaggeration to say that almost every educated man in the United States of America has now become 'meteor-minded.'

The International Astronomical Union has also done much to encourage research in Meteors. No account of the subject would be considered complete without paying tribute to the work of the great pioneer, W. F. Denning. The work started by him is carried on by the B. A. A. in England. Very valuable information is published from time to time in the Research Notes and other sections of *Nature*, on meteoric astronomy. Some of the back numbers of this great weekly are a veritable mine of information about important meteoric showers and fire-balls. Excellent statistical and observational work is being done by Hoffmeister in Germany: by Millman in Canada: by S. Orlov and I. Astapowitsch at Moscow, by Hideo Inouye, Kozira Komaki and members of the Astronomical Society in Japan.

Apart from the Leonids, many other periodic showers have been identified and studied, some of which have been traced, through old Chinese Annals, to as early a time as 687 B.C. Many of these showers are derived from the débris of matter revolving in the orbits of well-known comets; for example, the Leonids are associated with Tempel's Comet of 1866 I; the Andromedes with Biela's (last conspicuous shower occurring in 1885, Nov. 27 in the Third Burmese War); the Perseids with Tuttle's 1862 III; the Lyrids with Thatcher's 1861 I; and the Giacobinids with that discovered by Giacobini-Zinner. It may be mentioned in passing that a grand shower of the Giacobinids was expected on October 9, last year, but it failed completely. Very likely it may occur on October 10, this year (1940.)

[A comprehensive list of notable meteoric showers by the present writer, published in the *Journal of the Osmania University* Vol. III 1935, may be consulted for further reference.]

The Society for Research on Meteorites established in 1933 with Headquarters at Denver, Colorado, U.S A., has given an immense impetus to the study of Meteorites, and has brought to light a number of Meteorite falls and finds that would otherwise have been lost to the scientific world.

Meteoric Matter in Space

The density of interstellar space is reckoned by Eddington to be about 10^{-30} grams per c.c. Apart from the matter constituting the stars the amount of loose material or 'dust' in the universe is by no means inconsiderable. C. C. Wylie estimates a fall per hour, of about a million meteors, visible to the naked eye, all over the world. The number of telescopic meteors swells up to several millions of millions per diem.

[Spectroscopic evidence deduced from the presence of stationary H and K lines of ionised calcium, and of D_1 and D_2 and other Sodium lines, not to speak of two titanium lines and a neutral potassium line, with several others still unidentified, in the spectra of stars moving with large velocities in the line of sight, supports the view that interstellar space is by no means so devoid of material particles as the older astronomers were led to believe.]

Harvard College Circular No. 317 estimates a fall of at least a million million meteors *per second* in the atmospheres of some of the brightest stars, as revealed by the presence of Cyanogen bands, Swan spectrum and Raffety bands in their spectra; and 'guarantees' to them an almost endless life, in spite of their prodigal expenditure of high temperature energy.

Lights of Meteors and Meteor Trains

The cause of the light of meteors is traced to their bombardment by air molecules, when the meteors plunge into our atmosphere with terrific speeds; light being emitted when matter thus evaporating from the body collides with air molecules away from the main body. (It may be pointed out here that repeating shower meteors and most sporadic ones also are found to move in their orbits with velocities somewhat less than parabolic speeds, viz. 42 kms. per sec. near the earth. They are thus members of the Solar system. Some sporadic meteors, on the other hand, have been observed to move with much higher velocities, signifying motion in hyperbolic orbits, and are therefore considered to be of intersteller origin.)

Thanks to the laboratory experiments of C. C. Trowbridge on nitrogen after-glow, and of Kaplan, Vegard and others on gaseous ionisation, we are beginning to understand the mechanism of the mysterious streaks often seen *near* the end of a meteor's passage through the E region of the ionosphere. Even meteors of such low visibility as those of 2·5 or even 3 m. have been observed by the present writer to display appreciably conspicuous and fairly persistent streaks. Great advancement of our knowledge in this direction will follow if means are devised to obtain *pure* spectra of meteor trains "uncontaminated," so to say, by those of the meteors themselves.

Meteorite Craters

An interesting feature of the study of meteors and meteorites is the subject of meteorite craters, discussed in a number of recent papers. About six such objects have been recognised definitely up till now. The first to attract general attention is that at Canyon Diablo in Arizona, U. S. A., about 4000 ft. in diameter, 570 ft. in depth; the encircling wall rising from

130 to 140 ft. above the level of the surrounding country. Barringer seems to have been the first to suspect its meteoritic origin. Since his time several thousands of pieces of nickeliferous iron, with traces of platinum and iridium (sure signs of meteoritic character) have been picked up, outside the circular boundary of the great depression—one piece as large as 1014 lbs in mass. Search, through drilling, for the buried remnant of the original meteorite (presumably a siderite) that produced the Crater and is suspected to lie beneath its southern rim, has resulted in the recovery of rich meteoritic material from a depth of 1340 ft.

The crater must have been formed by the *explosive* impact of a gigantic meteorite, possibly a small comet or a tiny asteroid that came dangerously near to the earth, in its orbital motion round the Sun, some 10 or 12 thousand years ago. The dread evinced by the Red Indians of the locality in avoiding approach to it, lends support to the notion that a knowledge of the grim phenomenon with its terrible fire-blast and terrific explosion must have been handed down to the present race from generation to generation. Though the angle of hit is reckoned to be about 45° with the horizon, the huge mass must have struck the ground with parabolic velocity, the energy thus released vapourising the bulk of the meteorite and a large amount of the ground hit by it; the recoil of the compressed vapours at once scooping out a round depression and scattering molten matter in all directions.

Craters more or less similar to that in Arizona have been discovered at Wabar in Arabia (two, in 1934), at Odessa, in Ector County, Texas, U. S. A. (one, in 1933), on the island of Ösel in the Baltic, in the Tunguska Valley, Siberia (10 in number), and at Henbury, Central Australia (about 12, in 1931). The Tunguska craters were formed from the impact of large meteorites that fell in that part of Siberia in the early morning of June 30, 1908. Coming at a small angle with the horizon, after impact, thev gave rise to a pillar of smoke and dèbris that shot up to a height of 12 miles and was seen from 250 to 300 miles, setting fire to 3000 square miles of forest and causing earthquake shocks that were recorded all over the world!

There is some reason to believe that the Henbury craters were formed at no very remote date in the history of human habitation in Australia, as the name "chindu chinna waru chingi yabu" (Sun walk fire devil road) given by the aborigines of the country to the locality, signifies some acquaintance with the explosive origin of these craters.

It may be pointed out that the danger of a future collision between our Earth and a diminutive asteroid or dwarfish comet is by no means an astronomical fiction, though the devastation likely to be caused by the catastrophe may easily be exaggerated. There have been several dangerously near approaches of such bodies within recent years. Lexell's Comet of 1770 came probably to within 1 or 1½ million miles of the Earth. Adonis (1936 CA) discovered by Delporte came within 1⅓ million miles; and Reinmouth's object discovered in 1937, with a period of only 2·1 years came as close as 4 or 5 hundred thousand miles.

The Earth has probably experienced a number of such hard hits in the course of its 2000 millions of years' existence, and bears only a few dents or

dimples, here and there, from these causes, on its extensive surface. We may therefore, safely assume that the human race need fear no annihilation from such cosmic bombardments. A more serious source of calamity is its love of 'scientific' warfare and aerial raids!

Several geologists, notably Dr. L. J. Spencer, and a few astronomers hold the view that the craters on the moon are also of meteoritic origin. Their great numbers and large areas are not regarded by these investigators as serious obstacles in the formulation of a workable theory.

Concluding Remarks

It will thus be seen that meteoric astronomy is a fascinating new science, with ample opportunities for research, to the inquiring mind The deflection of well-known meteoric swarms by the perturbing effects of the larger planets, resulting in the anomalous behaviour of meteoric showers, presents a tempting field of investigation to trained mathematicians. As a matter of fact it is a favourite pursuit with a number of well-known observers to calculate the orbits of meteor-swarms that give rise to unusually rich showers and of exceptionally bright fireballs. To experimental physicists, equipped with adequate apparatus and working on the same meteors in accordance with pre-arranged plans at suitable distances, the light of persisting meteor trains is a picturesque problem in atomic physics holding out hopes of a surer knowledge of the condition, character and composition of the inaccessible regions of our atmosphere. The visual observer in his nightly watches discovers newer and newer radiants or confirms old ones, noting every peculiarity of magnitude, colour, apparent speed or duration, concerning all meteors that pass under his survey. The explorer of meteorites unearths and classifies an ever increasing number of these wonderful bodies lying buried underground for perhaps hundreds of years; or saves fresh ones from being lost, through unawareness or neglect. The meteorologist finds in the apparition of meteors new means of unravelling some of the mysteries of the weather lying outside the domain of clouds, fogs and snow. The anthropologist, discovering relics of metallic iron buried with religious care in the tombs of pre-historic men learns, with a little chemical analysis, to fix with greater accuracy the date of discovery of iron-smelting in the past and its bearing on human civilization. More than all this, meteoric astronomy raises the spirit of earth-bound man to cosmic heights, under the open sky and inspires him with veneration for the Great Design revealed in the mechanism of the boundless Universe.

List of those who were present at the 11th session of the Indian Mathematical conference held at Hyderabad from 21st to 23rd Dec. 1939

Abdul Azeez Esq., B.A., B.T., Training College, Hyderabad.
S. K. Abhyankar Esq., M.A., Victoria College, Gwalior.
Ahmad Mirza Esq., C. E., Chief Engineer, Dt. Buildings, Hyderabad.
H. A. Ansari Esq., B.A., Registrar, Osmania University.
M. V. Arunachala Sastry Esq., M.A., L.T., Secunderabad.
S. M. Azam Esq , Principal, City College.
B. B. Bagi Esq., Karnatak College, Dharwar.
E. Banerji Esq., M.A., D. A. V. College, Cawnpore.
N. M. Basu Esq., D.Sc., Dacca University.
G. V. Bhagwal Esq., M.A., Karnatak College, Dharwar.
T. P. Bhaskaran Esq., M.A., Director, Nizamia Observatory.
C. R. Chaturvedi Esq., Agra University.
M. S. Godbole Esq., M.A., B.T., Sir Parshurambhan College, Poona.
K. R. Gunjikar Esq., M.A., B Sc., Royal Institute of Science, Bombay.
Hafiz Abdul Haq Esq., M.A., Islamia College, Lahore.
A. Hameed Esq., M.A., Government College, Lahore.
Hameed Ahmad Ansari Esq., B.A., Registrar, Osmania University.
Hari Shankar Esq., Anglo Arabic College, Delhi.
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THE INDIAN MATHEMATICAL SOCIETY

Statement of Accounts for the year 1938

Receipts	Rs.	A.	P.
Opening Balance:			
P. O. Cash Certificate	5600	0	0
Fixed Deposit	500	0	0
Current A/c.	342	2	2
Savings Bank I	2352	5	9
Do. II	517	2	2
Membership Fees:			
Life members	325	0	0
Ordinary	483	0	0
Concession	420	0	0
Subscriptions for Journals	655	0	6
Sale of Journals and reprints	123	2	0
Grants in aid:			
Bombay University	200	0	0
Osmania University	100	0	0
Madras University	150	0	0
Punjab University ...	150	0	0
Interest on F. D. and S. B.	1694	3	3
Total ...	13611	15	10

Expenditure	Rs.	A.	P.
Administration—			
Hon. Librarian	301	0	0
Treasurer	10	0	0
Secretary	135	0	0
Assistant Secretary	189	11	0
Joint Editor	18	1	0
Journal printing	1078	14	0
Periodicals	409	13	6
Bank Charges	3	7	0
Closing balance—			
1. S. B.	1846	4	10
2. S. B.	10	5	4
3. S. B.	5	0	0
C. A.	604	7	2
Fixed Deposit	9000	0	0
Total ...	13611	15	10

L. N. SUBRAMANIAM,

Honorary Treasurer.

Indian Statistical Conference, Madras—Mysore : 1940.

The third session of the Indian Statistical Conference was opened in Madras on 3rd January 1940 by H. E. the Governor of Madras. Mr. V. V. Giri, ex-Minister for Labour, in welcoming the delegates, stressed the scope of statistics in the service of the State. The Honorary Secretary of the Institute, Prof. P. C. Mahalanobis, gave an account of the work done in Crop Forecasting, Labour Statistics, Diet and Health Surveys, the use of the method of samples and some theoretical studies, by that Institute during the year. The Presidential address was delivered by Professor H. Hotelling, Professor of Economics, Columbia University who had specially come to India as a visiting Lecturer to the Statistical Institute. He stressed the importance of theoretical studies, and also stated that the many ways in which statistical methods and statistical data are capable of improving human life, through the natural and the social sciences, and in industry and economic activities of myriad kinds, are only beginning to be understood in spite of their truly impressive accomplishments. The chaste beauty and intellectual delights of the theory of statistical inference, regarded as the offspring of mathematics and inductive logic, are known at present only to a few devotees ; but this theory is bound in time to receive a wider appreciation and a higher valuation even apart from its practical usefulness in the form of applications. Such appreciation will secure for statistical methods and statistical theory the interest and support which are necessary to enable their full potentialities being realised in bringing new light and new vigour to every department of national life.

Along with the Science Congress, joint meetings were held for the discussion of crop cutting experiments, and of the application of group theory to statistical distributions. Sections were also devoted to discussions on Census problems and labour statistics. Several interesting papers were also read by statistical workers from Madras, Calcutta, Trivandrum and other centres. Excursions had been arranged to the Industrial Museum and other places of interest.

After three days' session, the Conference adjourned to Mysore where a special session of two days had been arranged. Several studies on statistical theory and applied statistics mostly by workers in the psychology, economics and mathematics departments of the Mysore University were described and the discussions which they evoked were both learned and elaborate. Professor K. B. Madhava led the discussion on analysis of trade disputes, trade unionism and absenteecism in Indian Labour. The method of solving certain difference—differential equations was described at some length. A. A. Krishnaswamy Iyengar's study on Random lines evoked great appreciation, and Mrs. K. N. Kamalamma's Paper on the analysis of medical examination reports in Mysore dealt in a thorough manner with the descriptive and analytical aspects of the statistics. "Industrial Location in Mysore" was the title of paper which applied several indices to the local conditions in Mysore. The papers on Psychological unit of measurement (Dr. M. V. Gopalaswamy), Factor analysis in Aesthetic judgment (N. S. N. Sastry) elicited considerable comments from the President who also described how an electrical device by R. M. Mallock had been used for solving normal equations in incredibly short time. Excursions to the beauty spots, and to the industrial establishments, in and around Mysore City had been arranged, which along with the arrangements for accommodation and hospitality were greatly appreciated by the delegates.

K. B. M.

ERRATA FOR VOLUME VII

Page	*line*	*for*	*read*
9	6 from the bottom	e_1 is	e_1 corresponding to $\sqrt{a^2+\gamma}$ is
...	5 from the bottom	first or second	second or third
13	14	T	T_n
62	10	*for* $\frac{1}{2}(a+b+c)\sqrt{h^2+r_0^2+\frac{1}{2}r_0(a+b+c)}$	*read* $\frac{1}{2}(a+b+c)\sqrt{h^2+r_0^2}+\frac{1}{2}r_0(a+b+c)$
...	12	$\sqrt{h^2+r_0^2+r_0}$	$r_0+\sqrt{h^2+r_0^2}$
...	30	OD_3	OD_2
109	15	$f\ S_{kl}$	$f_k\ S_l$
,,	16	$f_1c_1+f_2c_2+f_3c_3$	$f_1C_1+f_2C_2+f_3C_3$
,,	22	$\frac{\partial(C_1,\ C)}{\partial(x,\ y)}$	$\frac{\partial(C_1,\ C_2)}{\partial(x,\ y)}$
,,	3 from the bottom	$+2f_3=$	$+2f_3$
174	4	grams	gram

Author Index of Questions and Solutions

(The Numbers refer to the pages)

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